Module 3

Junior Secondary Mathematics

Shapes and Sizes

The Commonwealth of Learning

Science, Technology and Mathematics Modules

for Upper Primary and Junior Secondary School Teachers
of Science, Technology and Mathematics by Distance
in the Southern African Development Community (SADC)

Developed by
The Southern African Development Community (SADC)

Ministries of Education in:

- **Botswana**
- **Malawi**
- **Mozambique**
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In partnership with The Commonwealth of Learning

Published jointly by The Commonwealth of Learning and the SADC Ministries of Education.

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ISBN 1-895369-65-7

SCIENCE, TECHNOLOGY AND MATHEMATICS MODULES

This module is one of a series prepared under the auspices of the participating Southern African Development Community (SADC) and The Commonwealth of Learning as part of the Training of Upper Primary and Junior Secondary Science, Technology and Mathematics Teachers in Africa by Distance. These modules enable teachers to enhance their professional skills through distance and open learning. Many individuals and groups have been involved in writing and producing these modules. We trust that they will benefit not only the teachers who use them, but also, ultimately, their students and the communities and nations in which they live.

The twenty-eight Science, Technology and Mathematics modules are as follows:

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Module 1: *My Built Environment*
Module 2: *Materials in my Environment*
Module 3: *My Health*
Module 4: *My Natural Environment*

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Module 3: *Living Organisms' Environment and Resources*
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Module 3: *Tools and Materials*
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Module 2: *Fractions*
Module 3: *Measures*
Module 4: *Social Arithmetic*
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Module 2: *Number Operations*
Module 3: *Shapes and Sizes*
Module 4: *Algebraic Processes*
Module 5: *Solving Equations*
Module 6: *Data Handling*

A MESSAGE FROM THE COMMONWEALTH OF LEARNING

The Commonwealth of Learning is grateful for the generous contribution of the participating Ministries of Education. The Permanent Secretaries for Education played an important role in facilitating the implementation of the 1998-2000 project work plan by releasing officers to take part in workshops and meetings and by funding some aspects of in-country and regional workshops. The Commonwealth of Learning is also grateful for the support that it received from the British Council (Botswana and Zambia offices), the Open University (UK), Northern College (Scotland), CfBT Education Services (UK), the Commonwealth Secretariat (London), the South Africa College for Teacher Education (South Africa), the Netherlands Government (Zimbabwe office), the British Department for International Development (DFID) (Zimbabwe office) and Grant MacEwan College (Canada).

The Commonwealth of Learning would like to acknowledge the excellent technical advice and management of the project provided by the strategic contact persons, the broad curriculum team leaders, the writing team leaders, the workshop development team leaders and the regional monitoring team members. The materials development would not have been possible without the commitment and dedication of all the course writers, the in-country reviewers and the secretaries who provided the support services for the in-country and regional workshops.

Finally, The Commonwealth of Learning is grateful for the instructional design and review carried out by teams and individual consultants as follows:

- Grant MacEwan College (Alberta, Canada):
  General Education Courses
- Open Learning Agency (British Columbia, Canada):
  Science, Technology and Mathematics
- Technology for Allcc. (Durban, South Africa):
  Upper Primary Technology
- Hands-on Management Services (British Columbia, Canada):
  Junior Secondary Technology

Dato' Professor Gajaraj Dhanarajan
President and Chief Executive Officer

ACKNOWLEDGEMENTS

The Mathematics Modules for Upper Primary and Junior Secondary Teachers in the Southern Africa Development Community (SADC) were written and reviewed by teams from the participating SADC Ministries of Education with the assistance of The Commonwealth of Learning.

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TEACHING JUNIOR SECONDARY MATHEMATICS

Introduction

Welcome to *Shapes and Sizes,* Module 3 of Teaching Junior Secondary Mathematics! This series of six modules is designed to help you to strengthen your knowledge of mathematics topics and to acquire more instructional strategies for teaching mathematics in the classroom.

The guiding principles of these modules are to help make the connection between theoretical maths and the use of the maths; to apply instructional theory to practice in the classroom situation; and to support you, as you in turn help your students to apply mathematics theory to practical classroom work.

Programme Goals

This programme is designed to help you to:

- strengthen your understanding of mathematics topics
- expand the range of instructional strategies that you can use in the mathematics classroom

Programme Objectives

By the time you have completed this programme, you should be able to:

- develop and present lessons on the nature of the mathematics process, with an emphasis on where each type of mathematics is used outside of the classroom
- guide students as they work in teams on practical projects in mathematics, and help them to work effectively as a member of a group
- use questioning and explanation strategies to help students learn new concepts and to support students in their problem solving activities
- guide students in the use of investigative strategies on particular projects, and thus to show them how mathematical tools are used
- guide students as they prepare their portfolios about their project activities

How to work on this programme

As is indicated in the programme goals and objectives, the programme provides for you to participate actively in each module by applying instructional strategies when exploring mathematics with your students and by reflecting on that experience. In other words, you "put on your student uniform" for the time you work on this course.

Working as a student

If you completed Module 1…did you in fact complete it? That is, did you actually do the various Assignments by yourself or with your students? Did you write down your answers, then compare them with the answers at the back of the module?

It is possible to simply read these modules and gain some insight from doing so. But you gain far more, and your teaching practice has a much better chance of improving, if you consider these modules as a *course of study* like the courses you studied in school. That means engaging in the material—solving the sample problems, preparing lesson plans when asked to and trying them with your students, and so on.

To be a better teacher, first be a better student!

Working on your own

You may be the only teacher of mathematics topics in your school, or you may choose to work on your own so you can accommodate this programme within your schedule. Module 1 included some strategies for that situation, such as:

1. Establish a regular schedule for working on the module.

2. Choose a study space where you can work quietly without interruption.

3. Identify someone whose interests are relevant to mathematics (for example, a science teacher in your school) with whom you can discuss the module and some of your ideas about teaching mathematics. Even the most independent learner benefits from good dialogue with others: it helps us to formulate our ideas—or as one learner commented, "How do I know what I'm thinking until I hear what I have to say?"

It is hoped that you have your schedule established, and have also conversed with a colleague about this course on a few occasions already. As you work through Module 3, please continue!

Resources available to you

Although these modules can be completed without referring to additional resource materials, your experience and that of your students can be enriched if you use other resources as well. There is a list of resource materials for each module provided at the end of the module.

ICONS

Throughout each module, you will find the following icons or symbols that alert you to a change in activity within the module.

Read the following explanations to discover what each icon prompts you to do.

	**Introduction**	Rationale or overview for this part of the course.
	**Learning Objectives**	What you should be able to do after completing this module or unit.
	**Text or Reading Material**	Course content for you to study.
	**Important—Take Note!**	Something to study carefully.
	**Self-Marking Exercise**	An exercise to demonstrate your own grasp of the content.
	**Individual Activity**	An exercise or project for you to try by yourself and demonstrate your own grasp of the content.
	**Classroom Activity**	An exercise or project for you to do with or assign to your students.
	**Reflection**	A question or project for yourself— for deeper understanding of this concept, or of your use of it when teaching.
	**Summary**	
	**Unit or Module Assignment**	Exercise to assess your understanding of all the unit or module topics.
	**Suggested Answers to Activities**	
	**Time**	Suggested hours to allow for completing a unit or any learning task.
	**Glossary**	Definitions of terms used in this module.

CONTENTS

Module 3: Shapes and Sizes

Module 3
Shapes and sizes

Introduction to the module

This module is concerned with shape and size of two dimensional figures. This branch of mathematics is generally referred to as geometry.

Geometry belongs to one of the oldest branches in mathematics. People from the early days of civilisation have been interested in describing physical forms they observed in their environment and looking for similarities in sizes and shapes. Earliest records, on Babylonian clay-tablets, of geometrical activities date back to 3000 BC. The Greeks developed geometry to a great height culminating in the *Elements* of Euclid (*c* 300 BC). This single work had an enormous impact on the future development of geometry. It used an axiomatic deductive method to prove 465 propositions in plane and solid geometry. For many ages hardly anything was added to this great Greek classic.

Expansion of geometry took place again after developments in algebra (16^{th} century in Italy). Desargues (1593 -1661) and Pascal (1623 - 1662) explored a new field in geometry: projective geometry which did not get much attention at the time and was only taken up again in the early nineteenth century. Descartes (1596 -1650) and Fermat (1601 - 1665), at about the same time as Desargues and Fermat developed analytical geometry, a combination of geometry and algebra. Analytical geometry deals algebraically with geometric properties of figures and is a method of geometry. A theorem in geometry is phrased and solved as a corresponding theorem in algebra. The first two units in this module use this method to deal with questions which are basically geometric in nature i.e. dealing with points, lines and closed shapes formed by line segments.

In the nineteenth century a number of geometries, different from the classical Greek Euclidean geometry, had developed. All have as a unifying base the study of properties of shapes that remain unchanged when subjected to a group of transformation. For plane Euclidean geometry the group of transformation are the reflections, rotations, translations and dilatations (enlargements) (NCTM, 1993).

Aim of the module

The module aims at:

(a) reflection on your present practice in the teaching of basic coordinate geometry and transformation geometry

(b) enhancing your content knowledge so that you may set activities on the geometry topics to your pupils with more confidence

(c) making your teaching of coordinate and transformation geometry more effective by using a pupil centred approach and methods such as group discussion, discovery method, games and investigations.

(d) reflecting on assessment of investigative work

Structure of the module

This module is divided into four units. Unit 1 and 2 deal with analytical geometry: the method of using algebra in answering geometric questions. In the first unit you will be looking at how the position of an object can be fixed in relation to other objects. In unit 2 you will look at points that are positioned on straight lines and how to describe them. Unit 3 will be looking at movements, in mathematics called transformations, of points and objects. There are many possible movements possible but you will focus on translation, reflection, rotation and enlargement and how to present activities to your students on these topics. Unit 4 will relate the movements studied in unit 3 to algebra. Matrices can be used to describe the movements. The last unit places emphasis on investigative work and hence also looks at how investigative work is to be assessed.

Objectives of the module

When you have completed this module, you should be able to create a learning environment for your pupils for them to acquire knowledge on:

(i) fixing of the position of objects

(ii) the Cartesian coordinate system as a way to describe position of points

(iii) the Cartesian coordinate system to describe lines and relationships between lines

(iv) transformation geometry in the plane

(v) matrices to describe transformation in the plane

using a pupil centred method in which pupils are actively involved in exploring the Cartesian plane and transformations and appropriate assessment procedure for investigative work.

Unit 1: Coordinate geometry I

Introduction

In this unit you will study ways to describe the position of a point. One widely used method is the use of Cartesian coordinates. The use of coordinates is commonly ascribed to René Descartes (1596 - 1650), although it seems that the notation of a coordinate system played little, if any, part in his work. The rectangular coordinates are named after Descartes: Cartesian coordinates. The use of rectangular coordinates to fix location of places is much older and dates back to Hipparchus (*c* 161 - 126 BC). The Cartesian coordinates will be used to fix position of points relative to an origin, calculate distances between points and to obtain coordinates of points positioned in a given way with respect to other given points.

Purpose of Unit 1

The purpose of this unit is to revise your knowledge on the use of Cartesian coordinates, to extend the concepts and to look at activities that could be used to introduce the concepts to pupils. The main aim is to help you to be a better teacher. When going through this unit (and others) reflect on how the material can be used in your classroom, whether it gives reasons to change your current classroom practice. Assignments are frequently related to trying out material in your classroom

We do hope that you and your pupils will benefit from this unit.

Objectives

When you have completed this unit you should be able to:

- state and illustrate four different ways to fix the location of a point
- list the concepts to be covered when introducing Cartesian coordinates
- list four points requiring specific attention when using the coordinate grid to plot points
- set activities to your class to introduce Cartesian coordinates
- use games in your class to consolidate Cartesian coordinates
- justify the use of games in the learning and teaching of mathematics
- distinguish between line, line segment and half line or ray
- distinguish between inductive and deductive methods in teaching/learning
- use an inductive or deductive discovery method with your pupils to find the coordinates of the midpoint of a given line segment
- use the Pythagorean theorem in finding distances between points

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A: Fixing position

Section A1: Exploring ways to fix position

1. Write down an outline of the lesson in which you introduce pupils to coordinates. How do you introduce coordinates to pupils, and how do you consolidate the concepts?

   Is the method you use different from the way you were introduced yourself to coordinates as a student? Justify differences and similarities.
2. Use your outline when working through this unit and attach it to your assignment.

The following is an outline of a classroom discussion on fixing position.

Objective: to make pupils aware that to fix the position of an object, you require (i) a reference point—you describe the position in relation to another (fixed) point; (ii) distances and/or directions

Teaching/Learning Aids: maps (country, Africa, universe), street maps of towns, plan of the school compound, plan of a house are displayed on the wall of the classroom

Discussion question for pupils

1. How would they describe to a stranger the location of
   a) their place in the classroom
   b) their house, post office, ... as related to the school
   c) sport field of the school, science block, ... as related to the classroom in which they are sitting
   d) the Head Master's office within the administrative building as related to the entrance of the building
   e) the village/town as positioned in the country
   f) the country as positioned within Africa
   g) the Earth as positioned in the universe (How would you direct aliens to the planet Earth?)
   h) chess pieces on a chess board/draft stones on a draft board

Follow up questions

2. a) What do you need to describe the position of something accurately?
   b) How accurate do you have to be?
   c) Can the position of an object be fixed using distances only? If yes: how many distances do you need to know? If No why not?
   d) Can the position of an object be fixed using directions only? If yes, how many directions do you need to know? If no, why not?
   e) How can directions be described?

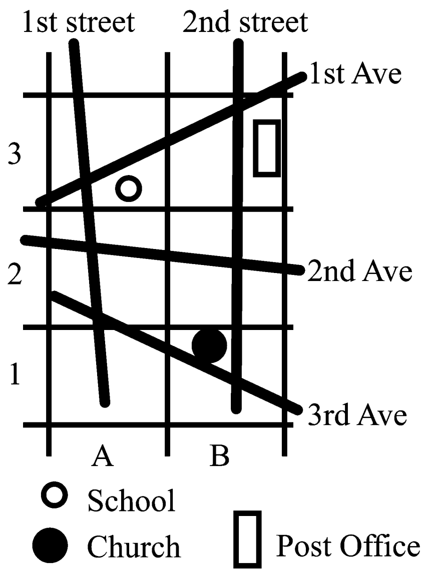
f) Is there a difference between fixing the position of a fly at a certain instance moving in the room and a fly sitting on the ceiling?

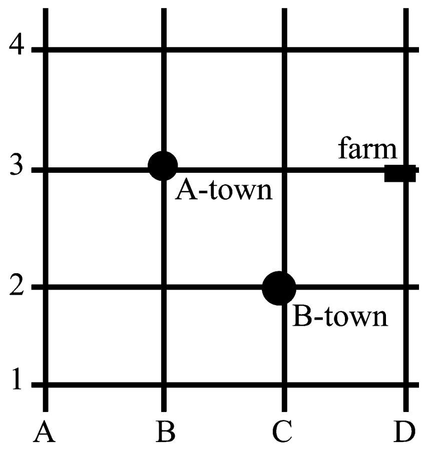
Self mark exercise 1

1. Answer the above ‘follow-up questions’ for yourself.
2. On geographical maps, longitude and latitude are used to fix the position of places. Explain the meaning of longitude and latitude. Illustrate with diagrams.

Check your answers at the end of this unit.

In the discussion pupils might come up with some of the following methods to fix position. If they did not, you might present the situations to them with the questions as illustrated below. Have the following situations ready as a hand-out for pupils or on an OHP transparency to show to the whole class.





Questions

a) Locating by giving the rectangle in which the street/place is found.

What is the location of the school?
Answer: in rectangle A3.

Where can we find 1^{st} Street?
Answer: in rectangles A1, A2 and A3

Discuss: In which situation is this way of locating accurate enough and suitable?

b) Locating points with two indicators.

The farm is located at D3

Which town is located at C2?

If C-town is at D4, draw/indicate where C-town is.

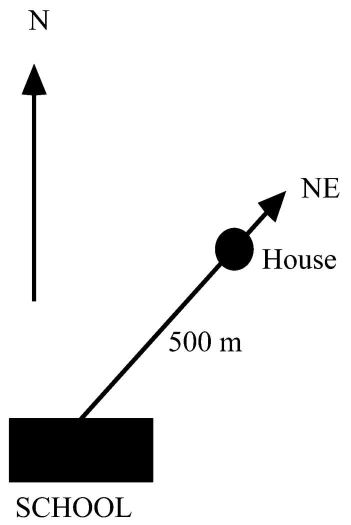
c) Giving the longitude and latitude of a place

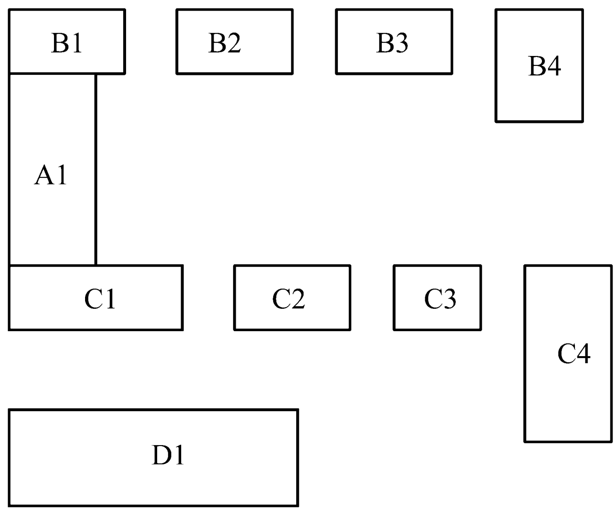
Find on a map the longitude and latitude of the capital of your country.

d) Locating points using distance and direction.

To reach my house from the school you have to walk 500 m from the front of the school in the direction North East (see diagram).

e) Locating by numbering (either single number or letters or combination). In a school compound the buildings/classrooms might be numbered B1, B2 etc.





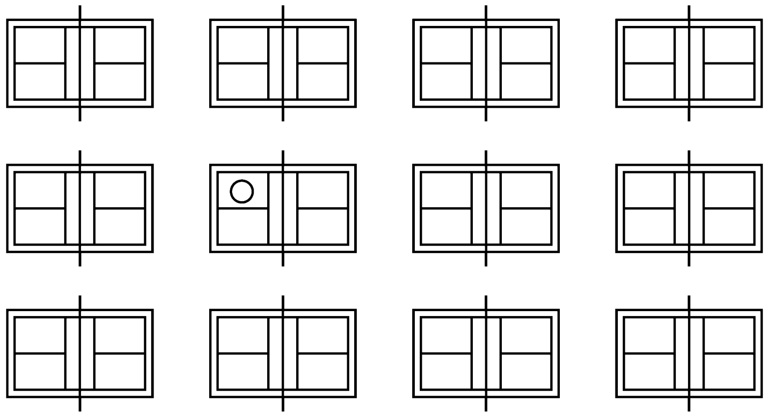
To check pupils understanding the following question can presented:

P1. On a sports ground there are 12 fields to play tennis. The layout of the field is presented to pupils on OHP transparency (or hand-out).

Modise is playing on field 8, indicated by o. She has to play her next game on field number 1.

Which field is field 1 according to you? Explain how you numbered the fields.

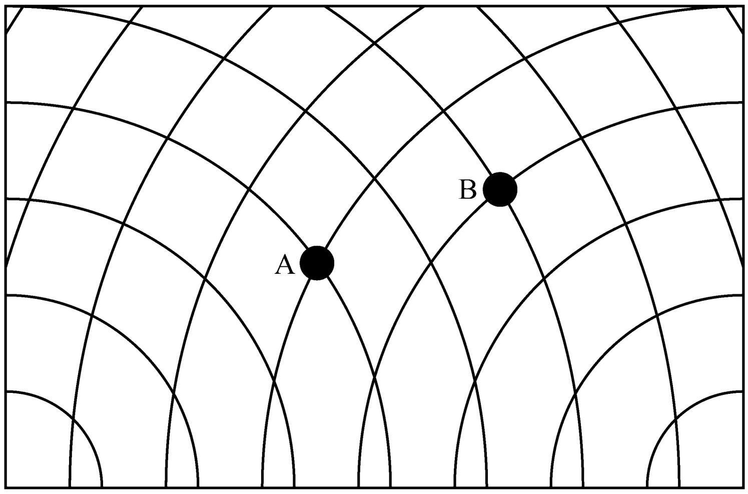
Modise did NOT go to the field you indicated as number 1. She thought field 1 to be another field than the one you indicated. To which field did she go?



P2. Use the following diagram to check pupils' understanding of coordinates in an application.

If A has coordinates (4, 5), what are the coordinates of B?

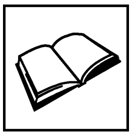
Plot C (6, 3)



Self mark exercise 2

1. Answer question **P1** and **P2**.
2. At what level of Bloom's taxonomy (knowledge, comprehension, application, analysis, synthesis or evaluation) do you place each of the questions **P1** and **P2**? Justify.

Check your answers at the end of this unit.

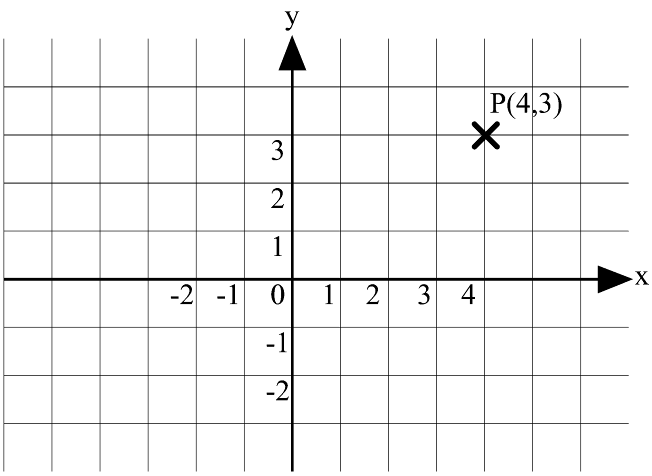


Section A2: Cartesian coordinate system

The position of a point in a plane can be fixed by two numbers, called the coordinates of the point. The numbers fix the position relative to a fixed point of origin.

concepts to be covered :

> *x*-axis, *y*-axis
> quadrants
> origin
> coordinates (abscissa, ordinate) = (*x*-coordinate, *y*-coordinate)
> scale



Plotting and reading of coordinates is not found difficult by pupils **when the coordinates involved are integers**. It is important to practice plotting points whose coordinates are rational numbers or decimal numbers. One potential problem must be cleared: frequently the axes are referred to as the horizontal and vertical axis. However this only applies to the teacher drawing on the vertical chalkboard! For pupils, working on a horizontal desk top, both axes are horizontal. Some teachers therefore use ‘across’ and ‘up’, before solving the problem by using the *x*-axis and *y*-axis.

Pay also attention to:

(i) axes are to be labelled,

(ii) the origin O is to be indicated,

(iii) along the axis a ‘start’ of the scale is to be indicated

(iv) points are indicated by X (meaning that the point is the intersection of the two short line segments)

(v) the common pupil’s error reading or plotting (x, y) as (y, x)

Consolidation activities to practice *plotting of points and reading of Cartesian coordinates:*

A1 Make various shapes by plotting points on the coordinate grid and joining them.

For example:

On a coordinate grid plot the following points, joining them in the order listed.

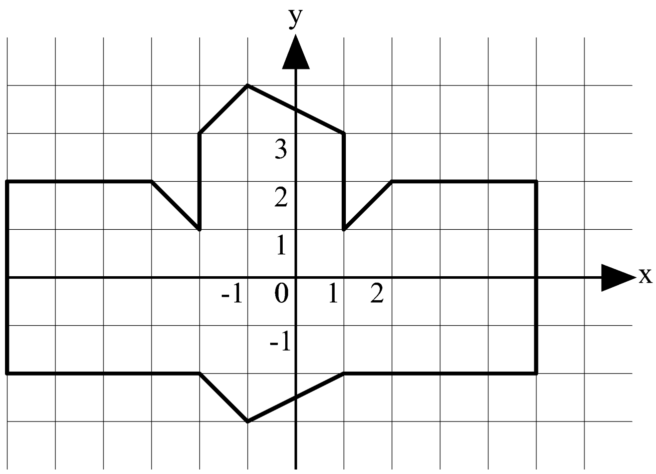
(0, 3), (1, 5), (1, 6), (2, 7), (3, 6), (5, 6), (6, 7), (7, 6), (8, 3), (6, 1), (2, 1), (0, 3)

Now join (3, 2) to (5, 2) and (4, 2) to (4, 3). Finally mark the points (3, 4) and (5, 4).

The activity allows for differentiation. The above example uses positive whole numbers (1^{st} quadrant) only as coordinates. Using fractional, decimal coordinates and coordinates in all the four quadrants can make the question more challenging for higher attainers.

A2 Read the coordinates in the order required for a given shape to be drawn.

Write down the instructions you would give to your friend to draw the illustrated diagram on a coordinate grid.



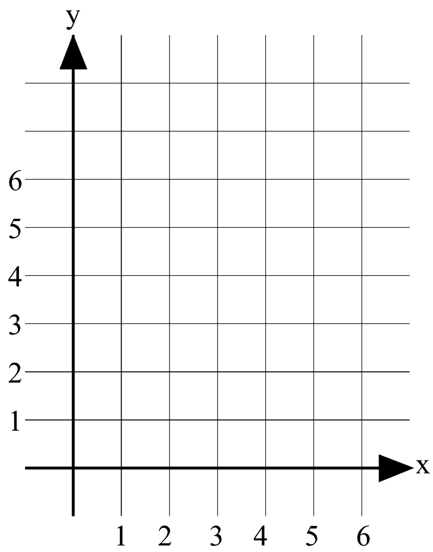
A3 Pupils make their own shape and write down the sequence of coordinates to be joined to give the shape they designed. Their sequence of coordinates can be given to another pupil to plot. The variety of shapes can make an attractive classroom display, especially when pupils add some colour to their designs.

A4 Game: Three in a line.

Objective: Reinforce plotting of points

2 players, coordinate grid, one dice.

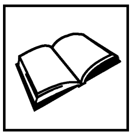
Players take turns in throwing the dice twice, first throw is the x-coordinate, second throw is the y-coordinate. Players mark points on the grid in different colours. The first player to get three points in a straight line wins the game.



Variation: Four in a line

Use a coordinate grid with four quadrants and four dice, two red (negative numbers) and two blue (positive numbers). Players select any two dice from the four and rolls the two chosen dice. The score on the dice determines the coordinates of the point.

The winner is the first player to obtain four points in a line, but not all points are allowed to be in the same quadrant.



Advantages of using games

Games can

1. ***Develop a positive attitude towards mathematics.***
   Pupils need to experience: success, excitement, satisfaction, enthusiasm, self-confidence, interest, enjoyment and active involvement. Few media are more successful than games in providing all of these experiences.
2. ***Consolidate mathematical concepts, facts, vocabulary, notation.***
   Concepts, facts, vocabulary, mathematical notation need consolidation. The traditional drill and practice episodes in a lesson are not, in general, very motivating, while a game like environment might consolidate the concepts in an enjoyable and motivating way. In particular games can be used to consolidate mathematical facts, vocabulary and notation.
3. ***Develop mental arithmetic skills.***
   Despite the fact that calculators are a tool in the learning of mathematics, this does not dismiss the need for pupils to know basic number facts and approximate sums, differences, products and quotients. Games can address specifically these important aspects for natural numbers, integers, (decimal) fractions and percent. For example a set of dominoes can be designed to consolidate the equivalence of fractions, conversion of fractions to percent or the four basic operations with integers.

4. ***Develop strategic thinking.***
   Games can encourage pupils to devise winning strategies. Can the person (i) playing first (ii) playing second always win? What is the 'best' move in a given situation?
   For example the game of noughts and crosses. Is there a winning strategy?

5. ***Promote discussion between pupils and between teacher and pupil(s).***
   When certain games are used in the mathematics class as a learning activity, there is a need for discussion: What mathematics did you learn? Is it a good game? Can it be improved?—are some of the questions to look at.

6. ***Encourage co-operation among pupils.***
   Some games require a group playing against another group or a pair of pupils against another pair. Such games can enhance co-operation among the group or the pupils paired. They are to co-operate in order to 'win' the game.

7. ***Contribute to the development of communication skills.***
   In a game the rules need to be explained. Pupils can explain the rules to others orally, formulate rules in writing, describe strategies used to each other—activities enhancing communication.

8. ***Stimulate creativity and imagination.***
   If pupils have been playing a game for some time they can be encouraged to make a similar new game for themselves or for younger brothers and sisters. Pupils frequently devise new rules to add to or to replace the basic rules to make the game more challenging to them once the basic rules have been mastered. Pupils can also be challenged to devise variations and extensions to the game. These activities call on pupils' creativity and imagination.

9. ***Serve as a source for investigational work.***
   Games can form a source for investigational work by analysing the game and answering questions such as: Is there a best move? Can the first player always win? How many possible moves are possible? Is it a fair game? What is the maximum score I can make? What would happen if ...? This can lead to looking at simpler cases first, tabulating results, making conjectures, testing hypotheses i.e. investigational work.

Unit 1, Practice activity 1

1. Use the outline you wrote at the beginning of this unit. After you have gone through this unit and tried out some of the activities in your class, would you make changes to your original outline? Justify changes you would introduce or justify that you stay with the method you have outlined from the start.
2. Set five questions to test pupils understanding of fixing location.
3. When introducing Cartesian coordinates, would you restrict the introduction to points in the first quadrant only or would you cover all four quadrants at once? Explain and justify.
4. a) Make three different sets of coordinate sequences which, when joined in order, will give a 'picture'. One should be for lower attainers, the other for average and higher attainers in your class. Give them to your pupils to consolidate plotting of points given in coordinate form. Write an evaluative report.

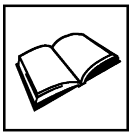
   b) Design a game for the consolidation of Cartesian coordinates. Try it out with your pupils and write an evaluative report.
5. Try out some of the suggested activities to consolidate the concept of Cartesian coordinates.

   Justify the activity you have chosen and write an evaluative report.

   Some questions you might want to answer could be: What were the strengths and weaknesses of the activity? What needs improvement? How was the reaction of the pupils? What did your learn as a teacher from the lesson? Could all pupils participate? How did you cater for the wide attainment range in your class? Were your objective(s) attained? Was the timing correct? Were you satisfied with the outcome of the activity?
6. Find out about **polar coordinates** and discuss the advantages and disadvantages as compared to Cartesian coordinates. Draw some curves given in polar coordinates.

Present your assignment to your supervisor or study group for discussion.

Section B: Midpoints and distances

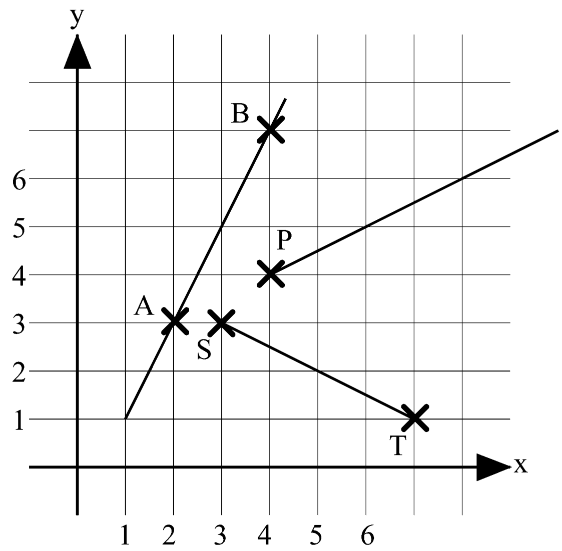


Section B1: Midpoint of a straight line segment

Points, lines, segments are all mathematical concepts. They are abstract ideas developed within the field of mathematics and exist only in our minds. Any diagram, drawing, picture can represent the idea but is not the idea itself.

For proper understanding of the following you should distinguish between the following:

- line straight: connection through two points extending indefinitely in both directions
- line segment: part of a line between two points. A line segment has a finite length: the distance between its endpoints.
- half line or ray: a straight line extending indefinitely in one direction from a fixed point



The diagram gives a representation of the line through A and B, the half line with end point P and the line segment ST.

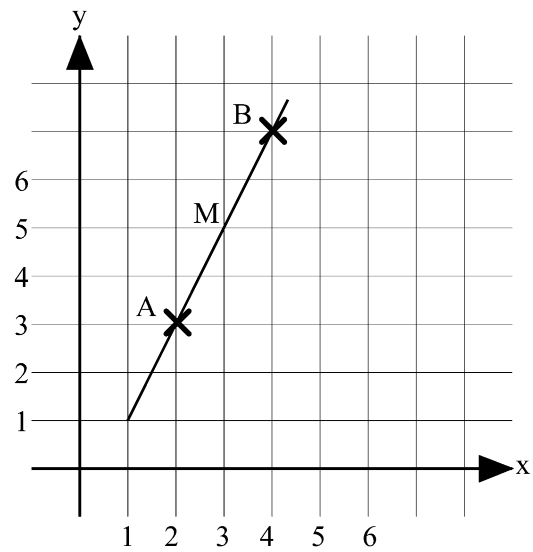
Section B2: Midpoints - worksheets for pupils

The following is an outline of a worksheet for pupils to discover how the coordinates of the midpoint of a line segment are related to the coordinates of its endpoints. An inductive method is used: from special cases pupils are to discover a pattern and suggest a relationship. The outcome of an inductive method is always ‘probable or likely knowledge’. It is NOT a proof—only a conjecture which might or might not be correct.

Pupils are expected to work in groups and discuss their work with each other.

Worksheet 1

On the coordinate grid the points A(2, 3) and B(4, 7) have been plotted. The midpoint M of the line segment AB has coordinates (3, 5) as can be obtained from the grid. M is midway between the endpoints. The distance AM and BM are equal.



a) Plot the following points on a coordinate grid and write down the coordinates of the midpoint

A	B	M
(0,1)	(0,7)	
(2,2)	(2,4)	
(1,4)	(7,4)	
(0,5)	(4,7)	
(0,8)	(4,0)	
(3,6)	(3,2)	
(1,1)	(5,3)	
(2,7)	(6,1)	
(4,0)	(0,8)	
(5,5)	(0,0)	

b) How are the coordinates of M related to the coordinates of A and B?

c) Without plotting the points write down the coordinates of the midpoint of the line segment with endpoints
(28,4) and (8, 10) (4, 16) and (36, 12) (32, 45) and (78, 155)

d) Investigate for points with one or both coordinates negative. Make a table similar to the one in a). Does your rule still work?

A	B	M
(0, $^-1$)	(0, 7)	
(2, 2)	(2, $^-4$)	
($^-1$, 4)	($^-7$, 4)	
(0, $^-5$)	($^-4$, 7)	
(0, $^-8$)	($^-4$, 0)	
($^-3$, 6)	($^-3$, $^-2$)	
(1, 1)	(5, $^-3$)	
(2, $^-7$)	(6, 1)	
(4, 0)	(0, $^-8$)	
($^-5$, $^-5$)	(0, 0)	

e) Without plotting the points, write down the coordinates of the midpoint of the line segment with endpoints

(i) ($^-18$, 4) and (6, $^-10$)

(ii) (24, $^-60$) and ($^-34$, $^-12$)

(iii) (36, 45) and ($^-78$, $^-135$)

f) (i) The midpoint of AB is (2, 3) and point A is at (1, 1). What are the coordinates of B?

(ii) The midpoint of AB is ($^-2$, 3) and point A is at (1, 1). What are the coordinates of B?

(iii) The midpoint of AB is (2, $^-3$) and point A is at (1, 1). What are the coordinates of B?

(iv) The midpoint of AB is ($^-2$, $^-3$) and point A is at (1, 1). What are the coordinates of B?

(v) Tabulate your results.

A	M	B
(1, 1)	(2, 3)	
(1, 1)	(${}^-2$, 3)	
(1, 1)	(2, ${}^-3$)	
(1, 1)	(${}^-2$, ${}^-3$)	

Can you find a rule? Check your rule for some more cases given coordinates of the midpoint and point A.

Self mark exercise 3

1. Work through the pupils' worksheet 1.

Check your answers at the end of this unit.

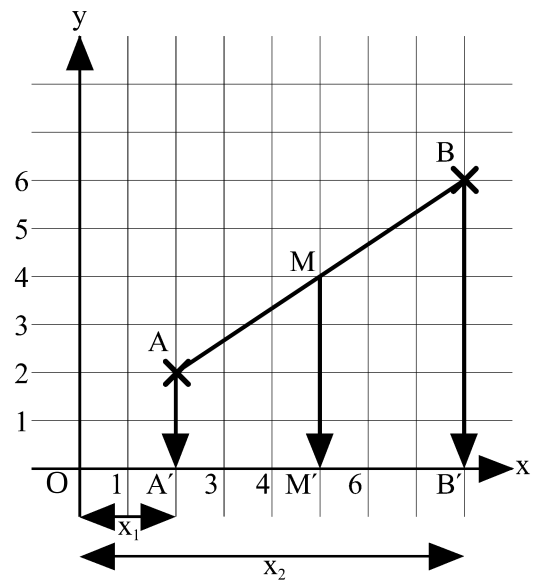
Worksheet 2

Worksheet for higher attainers. Deductive discovery method is used: from previous knowledge new knowledge is developed. Deductive methods lead to ‘sure’ knowledge, it proves the result.

Pupils are expected to work in groups and discuss their work with each other.

a) The diagram illustrates the line segment AB with midpoint M. Let us take A and B as some general points and derive the coordinates of the midpoint M.

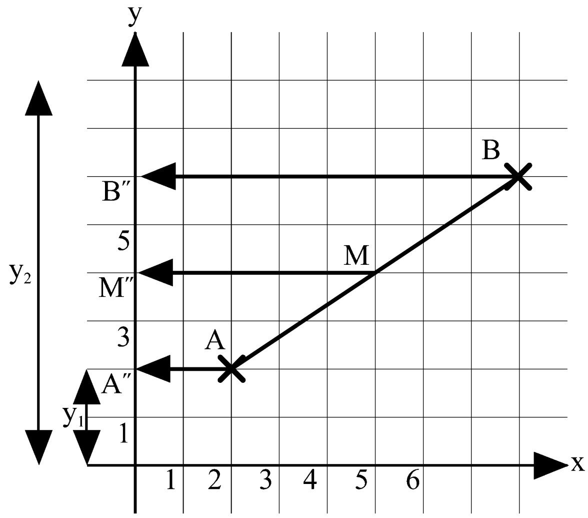
   Let A(x_1, y_1) and B(x_2, y_2)



   The x-coordinate of A, x_1, is represented in the diagram by OA′.
   Hence x_1= OA′

   Complete the following to find the x-coordinate of M, the midpoint of AB.
   A′B′ = OB′ – OA′ = x_2 – ....
   A′M′ = $\frac{1}{2}$ A′B′ = $\frac{1}{2}(x_2 - ...)$
   OM′ = x_M = OA′ + A′M′ = x_1 + $\frac{1}{2}(x_2 - ...)$
   (expand and simplify) = ...
   Hence the x-coordinate of the point M: x_M = ...

b) Repeat the above but now for the y-coordinates. Use the following diagram and find an expression for OM″ = y_M, the y-coordinate of the midpoint.



Complete the following to find the y-coordinate of M, the midpoint of AB.

A″B″ = OB″ - OA″ = ...

A″M″ = $\frac{1}{2}$ A″B″ = ...

OM″ = y_M = OA″ + A″M″ = ... (expand and simplify)

= ....

Hence the y-coordinate of the point M: y_M = ...

c) Summarise your finding:
If A(x_1, y_1) and B(x_2, y_2) are the coordinates of the endpoints of line segment AB then the coordinates of the midpoint M of AB are: (.... , ...)

d) Calculate the coordinates of the midpoint of the line segments with, as endpoints, the points with the following coordinates:

(i) (0, $^-3$) and (5, 0)

(ii) ($^-1\frac{1}{2}$, 4) and (0, $^-2\frac{1}{2}$)

(iii) (3.4, $^-6.7$) and ($^-8.3$,17.2)

e) The coordinates of the midpoint of the line segment with coordinates of its endpoints (x, $^-5$) and (0, $^-3$) are (2, y). Calculate the value of x and y.

f) The coordinates of the midpoint of the line segment with coordinates of its endpoints (x, 2.5) and ($^-6$, 4) are ($^-3\frac{1}{2}$, y).
Calculate the value of x and y.

g) The coordinates of the midpoint of the line segment with coordinates of its endpoints ($^-3.4$, 2.5) and ($^-6$, 4) are (x, y). Calculate the value of x and y.

Self mark exercise 4

1. Work through the pupils' worksheet 2.
2. Using a similar deductive method as in worksheet 2 show that the coordinates of the point P dividing the line segment AB into the ratio AP : PB = 1 : 3 are given by

   P$\left(\frac{3x_1 + x_2}{4}, \frac{3y_1 + y_2}{4}\right)$, where the coordinates of the endpoints of the line segment are A(x_1, y_1) and B(x_2 ,y_2).
3. Using a similar deductive method as in question 2 show that the coordinates of the point P dividing the line segment AB into the ratio AP : PB = p : q are given by

   P$\left(\frac{qx_1 + px_2}{p+q}, \frac{qy_1+py_2}{p+q}\right)$,where the coordinates of the endpoints of the line segment are A(x_1, y_1) and B(x_2 ,y_2).

Check your answers at the end of this unit.

Unit 1, Practice activity 2

1. You learned to distinguish between line segment, half line (or ray) and line. Do you feel these differences are important to pupils, i.e. would you introduce the concepts to pupils or not? Justify.
2. You worked through the suggested worksheet W1 and W2 for pupils to discover how the coordinates of the midpoint of the line segment is related to the coordinates of its end points. Will it be appropriate for use in your class? Justify your answer.

   Does it need improvement? Does it cater for all pupils or is a version, more guided, for lower attainers needed? If yes, develop such a worksheet.
3. Try out the (improved) version(s) of the worksheet in your class and write an evaluative report.
4. Both worksheets need to be supported by consolidation work. Write for both worksheets 10 consolidation questions. Justify why you included each question in the exercise and show the working you expect from pupils working the exercise.

Present your assignment to your supervisor or study group for discussion.

Section B3: Distance between two points with coordinates (x_1, y_1) and (x_2, y_2)

The distance between two points given their coordinates is an application of the Pythagorean theorem, and should be introduced after the Pythagorean theorem has been covered by the pupils.

The assumed and prerequisite knowledge for this section is therefore the Pythagorean theorem.

Self mark exercise 5

1. Give a concise formulation of the Pythagorean theorem in terms of length of the sides of a right-angled triangle.
2. Give a concise formulation of the Pythagorean theorem in terms of the area enclosed by squares on the sides of a right-angled triangle.
3. a) If $d^2 = 9$, find the value(s) of d.

   b) If $d = \sqrt{9}$, find the value(s) of d.

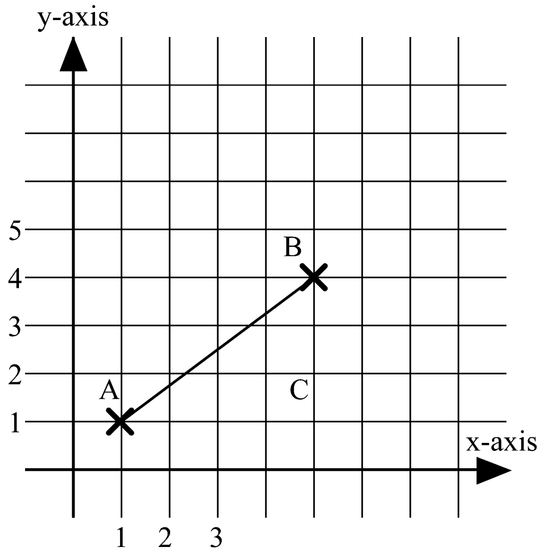
Check your answers at the end of this unit.

As pupils are assumed to be familiar with the Pythagorean theorem, the ‘distance formula’ can be developed through (i) teacher guided class discussion (ii) guided worksheet.

An outline of a worksheet follows here. Pupils are expected to work in groups and discuss their work with each other.

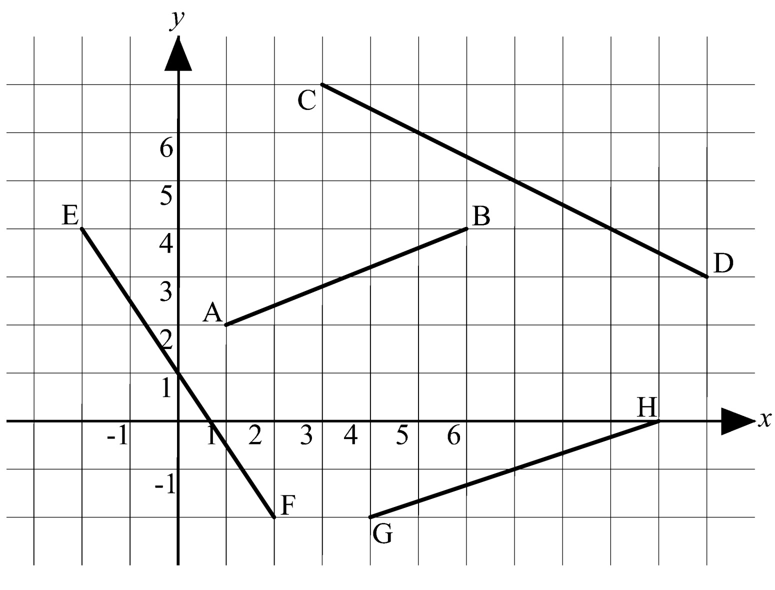
Worksheet 3

a) On a map two places A and B are situated at (1, 1) and (5, 4) respectively (distances in km). A road is to be constructed connecting the two places directly. What will be the length of the road?



AB is the hypotenuse of the right angled triangle ACB.
What is the length of AC?
What is the length of BC?
Apply the Pythagorean theorem to find the distance AB.

b) Calculate the lengths AB, CD, EF and GH, correct to 1 decimal place.

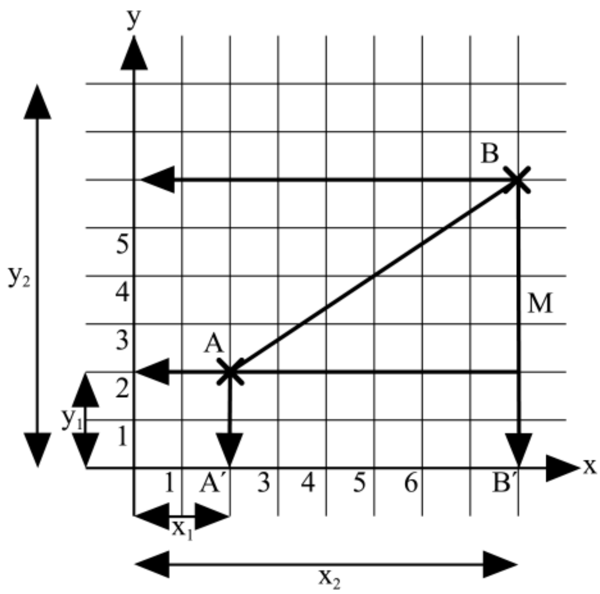


c) Plot these pair of points with given coordinates on squared paper and calculate the length of the line segment joining them

(i) O(0, 0) and P(8, 6) (ii) Q(1, 2) and R(-4, 7)

(iii) S($^{-}6$, 8) and T($^{-}1$, $^{-}1$)

d) Let A(x_1, y_1) and B(x_2, y_2)



The x-coordinate of A, x_1, is represented in the diagram by OA′.
Hence x_1 = OA′

AC = A′B′ = OB′ - OA′ = x_2 – .... (i)

BC = ... (ii)

Applying the Pythagorean theorem in triangle ACB: $AB^2 = AC^2 + BC^2$

Substitute for AC and BC the expressions obtained in (i) and (ii) respectively

$AB^2 = (x_2 - ....)^2 +$ ....

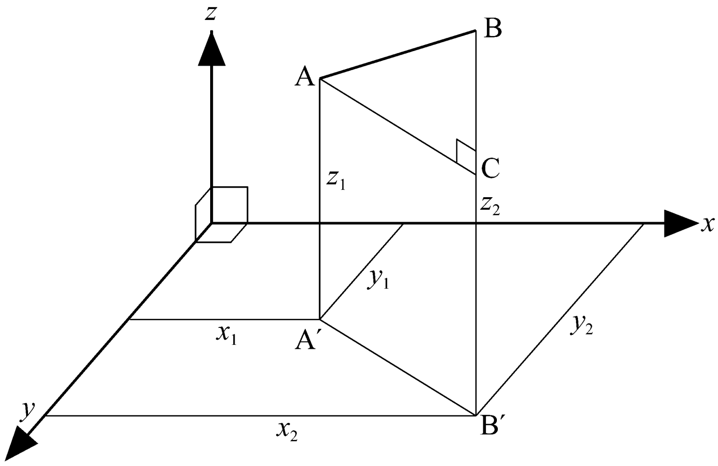
AB = ... (or AB = - ...., not acceptable as distances are positive).

In summary if the coordinates of two points are A(x_1, y_1) and B(x_2, y_2) then the distance between the two points is ......

The notation used in several books for the distance between two points A and B is |AB|, meaning the **modulus** or length of the line segment with end points A and B. Since "modulus" sounds a lot like the word modulo (as in q mod 4 = 1), it's use for the length of a line segment is not recommended for secondary students.

Self mark exercise 6

1. Work through pupils' worksheet 3.
2. The outcome of worksheet 3 is the distance formula for the distance between two points A(x_1, y_1) and B(x_2, y_2).
   $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
   Explain why this can also be written as $|AB| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
3. a) In 3 dimensions you will need three coordinates to fixed the position of a point. What is the distance between two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2)? First find A´B´ (= AC), next apply the Pythagorean theorem in triangle ACB to find AB.



   b) What are now the coordinates of the midpoint M of the segment AB?
4. If A(0, 0) and B(b, 0) find the coordinates of the point(s) C such that
   a) triangle ABC is isosceles
   b) triangle ABC is equilateral

Check your answers at the end of this unit.

Unit 1, Practice activity 3

1. In this unit you met with two distinct formulations of the Pythagorean theorem: one in terms of areas of squares on the sides of the right-angled triangle and the other in terms of lengths of the sides.
   a) Which of these two do you use in setting activities that lead the pupils to the discovery of the theorem?
   b) Describe the method you use to move from the Pythagorean theorem form discovered by the pupils to the other form.
2. Analytical or coordinate geometry is a method that can be used to prove geometrical properties that might have been obtained in a geometrical way (using symmetry or congruency).
   a) Design a worksheet for pupils to prove using coordinate geometry the following geometrically known properties:
      (i) the diagonals in a rectangle are equal in length
      (ii) the diagonals in a rectangle bisect each other
   b) Try out the worksheet in your classroom
   c) Discuss with your pupils which method has their preference: the geometrical or the analytical way to show the property
   d) Write an evaluative report on the lesson, including reactions of pupils.

Present your assignment to your supervisor or study group for discussion.

Summary

This unit introduced some modern classroom techniques for starting your pupils in analytic geometry. Likely these methods are different from ones you were taught as a child. For one thing, they employed the relatively "sloppy" inductive method; rigorous proof was left out, or was set aside for just the higher attainers in your class. For another thing, did you notice that Euclid's geometry is not taught in this course at all? In effect, Descartes has supplanted Euclid. Traditionalists bemoan Euclid's passing, but the change simply reflects the relative power of Cartesian methods for solving geometrical puzzles in everyday life. In like vein, students at this age have a better grasp of inductive reasoning. For purists who say that the more formal logic of deduction and proof is necessary for endeavours like law, the facetious answer is to teach it in law school! But more realistically, it is better used as a learning medium when most students are a few years older.

Unit 1: Answers to self mark exercises

Self mark exercise 1

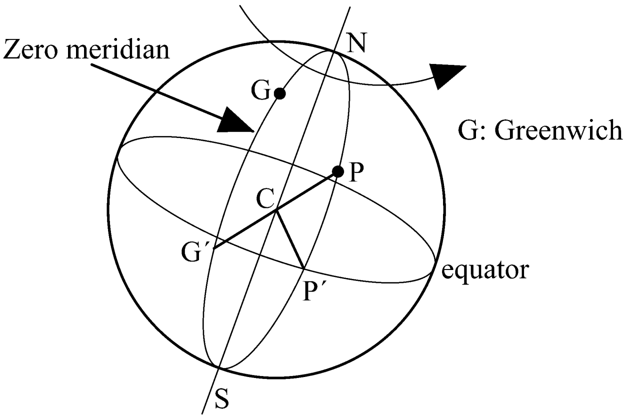
1. There are various ways to describe the position of a point accurately: using distance(s) and/or direction(s) from (a) fixed point(s).

   e.g. a point P in a plane can be fixed by any of the following

   - the Cartesian coordinates of P with respect to a fixed point
   - distance and direction (bearing or angle made with a fixed line e.g. x-axis) from a fixed given point (polar coordinates)
   - directions from two fixed given points
   - distances from three given fixed points (distances from two points does not uniquely fix the position/it allows for two possibilities).

2. Latitude - angular distance of a point on the earth's surface measured from the equator along the meridian passing through that point

   Longitude - angular distance of a point on the earth's surface measured along the equator between the prime (zero or Greenwich) meridian and the meridian through the point



   The latitude of the place P is the size of angle PCP′, where C is the centre of the earth and P and P′ are on the meridian through P, with P′ on the equator.

   The longitude of P is the size of angle G′CP′

Self mark exercise 2

1. P1 bottom far right/bottom far left P2 B(6, 4)
2. P1 analysis P2 application/analysis

Self mark exercise 3

1. Worksheet W1

   a) (0, 4); (2, 3); (4, 4); (2, 6); (2, 4) ; (3, 4); (3, 2); (4, 4); (2, 4); (2.5, 2.5)

   b) x-coordinate of the midpoint is the average of the x-coordinates of the endpoints. y-coordinate of the midpoint is the average of the y-coordinates of the endpoints.

   c) (18, 7); (20, 14); (55, 100)

   d) $(0, 3)$; $(2, {}^{-}1)$; $({}^{-}4, 4)$; $({}^{-}2, 1)$; $(-2, {}^{-}4)$; $({}^{-}3, 2)$; $(3, {}^{-}1)$; $(4, {}^{-}3)$; $(2, {}^{-}4)$; $({}^{-}2.5, {}^{-}2.5)$

   e) $({}^{-}6, {}^{-}3)$; $({}^{-}5, {}^{-}36)$; $({}^{-}21, {}^{-}45)$

   f) (i) $(3, 5)$; (ii) $({}^{-}5, 5)$; (iii) $(3, {}^{-}7)$; (iv) $({}^{-}5, {}^{-}7)$

   g) $(3, 5)$; $({}^{-}5, 5)$, $(3, {}^{-}7)$, $({}^{-}5, {}^{-}7)$

Self mark exercise 4

1. Worksheet W2

   a) $A'B' = x_2 - x_1 \quad A'M' = \frac{1}{2}(x_2 - x_1)$

   $OM' = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_2 + x_1) \quad x_M = \frac{1}{2}(x_1 + x_2)$

   b) As (a) replacing x by y.

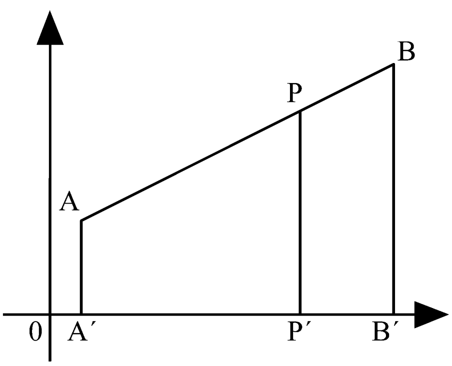
   c) $M\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(x_1 + x_2)\right)$

   d) $(2.5, {}^{-}1.5)$; $\left(\frac{{}^{-}3}{4}, \frac{{}^{-}3}{4}\right)$; $({}^{-}2.45, 5.25)$

   e) $x = 4, y = {}^{-}4$ f) $x = {}^{-}1, y = 3.25$ g) $x = {}^{-}4.7, y = 3.25$

3. In the diagram AP : PB = A′P′ : P′B′ = $p : q$

   $A'P' = \frac{p}{p+q} A'B' = \frac{p}{p+q}(x_2 - x_1)$



$$OP' = x_p = x_1 + \frac{p}{p + q}(x_2 - x_1) =$$

$$\frac{(p + q)\,x_1 + p(x_2 - x_1)}{p + q} = \frac{qx_1 + px_2}{p + q}$$

Similarly for the *y*-coordinate of the point P.

Self mark exercise 5

1. In a right angled triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two right angled sides.
2. The area of the square on the hypotenuse of a right-angled triangle is equal to the areas of the squares on the two right-angled sides of the triangle.
3. a) $^-3$, 3 b) 3 (NOT $^-3$)

Self mark exercise 6

1. Worksheet 3

   (a) 5 km; AC = 4, BC = 3, AB = 5

   (b) 5.4; 8.9; 7.2; 6.3

   (c) (i) 10 (ii) $5\sqrt{2}$ (iii) $\sqrt{106}$

   (d) $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

2. $(x_2 - x_1)^2 = x_2^2 - 2x_1x_2 + x_1^2$ and $(x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$ hence

   $(x_2 - x_1)^2 = (x_1 - x_2)^2$.

   Similarly for the expression in the *y*-coordinates, hence:

   $$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3. a. $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

   b. $\left(\frac{x_1 + \ + x_2}{2}, \frac{y_1 + \ + y_2}{2}, \frac{z_1 + \ + z_2}{2}\right)$

4. a. $(\frac{b}{2}, c)$ b. $(\frac{b}{2}, \frac{b}{2}\sqrt{3})$ *or* $(\frac{b}{2}, \frac{^-b}{2}\sqrt{3})$

Unit 2: Coordinate geometry II

Introduction to Unit 2

This unit looks at ‘straight lines’. Some people query the word ‘straight’ as in their definition of a line a line, is always straight. If it is not straight it is not a line. The concept “straight line” seems obvious to many a person until asked to define exactly what is meant by it. Attempts to define a straight line include those by:

Plato (380 BC): “that of which the middle covers the ends” (an expression of the view from the eyes placed at either end, looking along the line).

Euclid (300 BC): “that which lies evenly with points on itself.”

Archimedes (225 BC): “of all lines having the same extremities the straight line is the shortest”. This is the source of the common definition: the shortest distance between two points.

This however does not solve our problems completely, as commonly a point is defined as “the intersection of two lines” i.e., a circular definition!

Purpose of Unit 2

The purpose of this unit is to revise your knowledge on gradients of straight lines, relations between gradients of lines that are parallel or perpendicular to each other and on equations of straight lines. The emphasis is on how you can present the concepts to the pupils in your class.

Geometrical properties are investigated using coordinate geometry.

Objectives

When you have completed this unit you should be able to:

- set learning activities to your pupils to cover the concepts (i) gradient or slope (ii) y-intercept (iii) x-intercept
- identify straight lines with negative, zero, positive and no gradient
- relate gradients with angles the line makes with the positive x-axis
- set learning activities to your pupils to discover that (i) parallel lines have equal gradients (ii) lines with equal gradients are parallel
  (iii) the gradients of perpendicular lines have as product $^{-}1$, provided no horizontal /vertical line is involved (iv) if the product of the gradients of two lines is $^{-}1$ the lines are perpendicular
- prove that (i) if two lines (not horizontal or vertical) are perpendicular the product of their gradients is $^{-}1$ (ii) if the product of the gradients of two lines is $^{-}1$ the lines are perpendicular
- plot graphs of straight lines given their equation
- use coordinate geometry in investigating geometrical properties

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

Section A: Straight lines

Section A1: Gradients of straight lines

Plotting points that satisfy a relationship of the form $y = ax + b$ and joining the points plotted will ALWAYS give a straight line.

Equations of the form $y = ax + b$ are equations of straight lines (but not all straight lines have equations of the form $y = ax + b$). They are called **linear** equations. The word **linear** contains the word **line**.

To plot the graph of the line with equation $y = ax + b$ you:

(i) find and tabulate the coordinates of three points on the line:

(ii) plot the points on a labelled coordinate grid. Use X (two small crossing lines) to indicate the position of the point.

(iii) join the points, extending the line at both sides.

(iv) label the line with its equation.

The following are elements for inclusion in a worksheet for pupils to develop the concept of gradient. Pupils are expected to work in groups and discuss their work with each other.

An equivalent term for 'gradient' is 'slope', common in North America and in textbooks from there.

Worksheet 4 (outline)

Objective: The pupils should relate 'steepness' and gradient ('stepsize' when moving 1 unit across) and be able to find gradient of lines (positive gradients only).

1. a) Using the four steps outlined draw on the same coordinate grid lines with the following equations
   (i) $y = x$ (ii) $y = 2x$ (iii) $y = 3x$ (iv) $y = 4x$

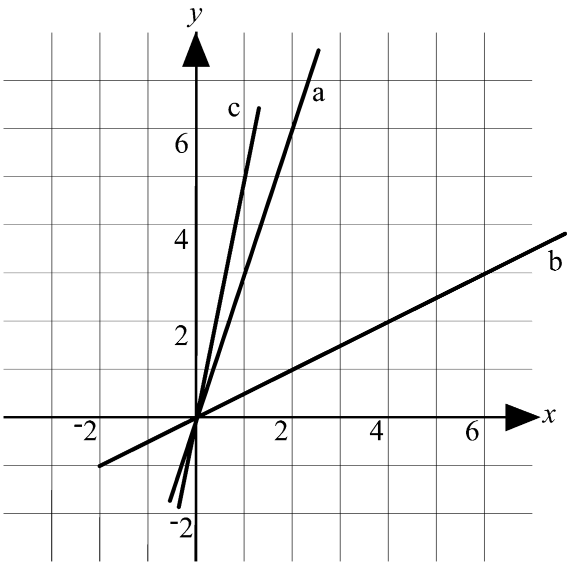
   b) Which line is the steepest?

   c) Which line is least steep?

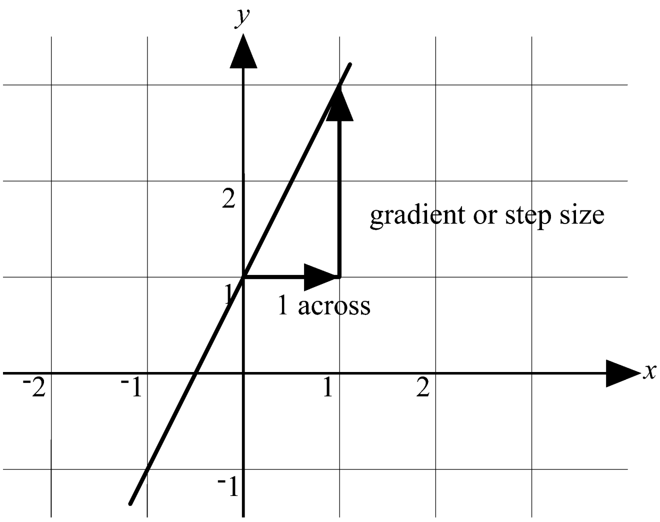
   d) Which part of the equation tells you how steep the line is?

2. Which line is the steepest in each of the following pairs?
   a) $y = 3x$ and $y = 5x$ b) $y = 4x$ and $y = x$
   c) $y = 3\frac{1}{2}x$ and $y = 3\frac{3}{4}x$ d) $y = 4.1x$ and $y = 4.11x$

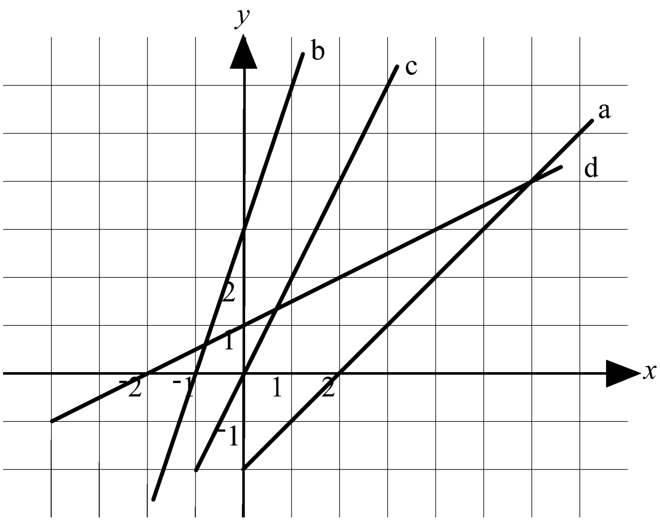
3. Which equation belongs to line a, b and c? Choose from
   $y = \frac{1}{2}x$, $y = 5x$ and $y = 3x$



The ‘steepness’ of a line is called its gradient, or slope, or step size. It tells you how high up the graph goes for ONE step across.



4. What is the gradient of the lines a, b, c and d?



5. a) The points P (2, 1) and Q(4, 5) have been plotted.

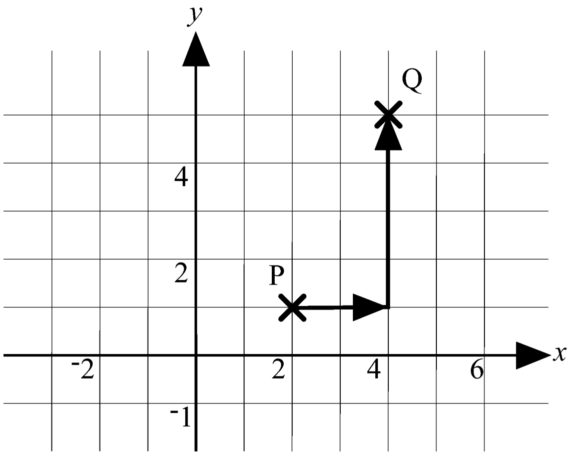
How many steps to the right and how many steps up will you move to move from P to Q?

Complete:

If you move .... steps to the right, you move .... steps up.

If you move 1 step to the right, you will move up .... steps.

What is the gradient of the line through P and Q?



b) Calculate the gradient of the lines passing through A and B if

(i) A(1, 2) and B(3, 8)

(ii) A(4, 1) and B(7, 7)

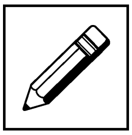
(iii) A($^-3$, $^-2$) and B(1, 10)

(iv) A($^-4$, 2) and B($^-1$, 6)

(v) A($^-2$, 4) and B(2, 8)

(vi) A(3, 1) and B(7, 1)

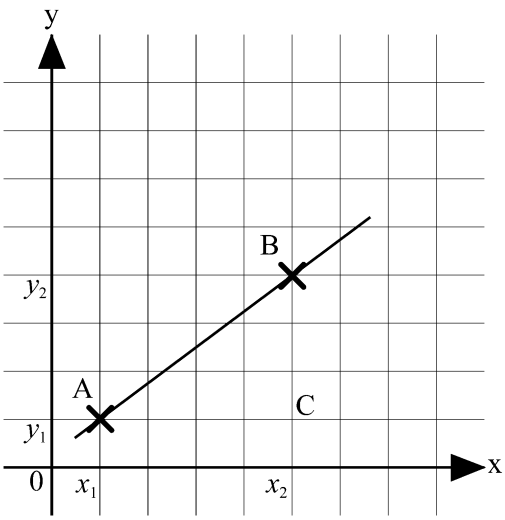
(vii) A(2, 1) and B(2, 5)



Self mark exercise 1

1. Work the suggested questions in worksheet 4.
2. Are ALL lines of the format $y = ax + b$? Justify your answer.
3. What is special about lines with gradient 0? What type of equation do these line have?
4. Which lines have NO gradient? What type of equation do these line have?
5. a) What is the gradient of the x-axis? What is its equation?

   b) What is the gradient of the y-axis? What is its equation?
6. Derive an expression for the gradient of a line through the points with coordinates A(x_1, y_1) and B(x_2, y_2).



   Express AC in terms of the x-coordinates of A and B, express BC in terms of the y coordinates of A and B. Hence find the gradient of AB.

7. a) Did you find in 6 that the gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$. Explain

   (i) why you can also write for the gradient $\frac{y_1 - y_2}{x_1 - x_2}$.

   (ii) why $x_1 \neq x_2$. What line do you have in case the two x-coordinates are equal?

   b) What trigonometric ratio of angle BAC is given by the gradient?

   c) Relate the tangent of the angle between the line and the positive x-axis to the gradient of the line.

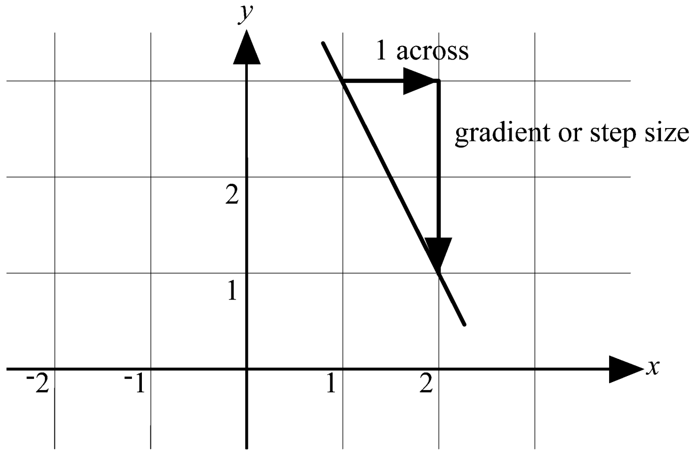
   d) Relate the size of the angle between the line and the positive x-axis (0°, acute, 90° or obtuse) to the gradient of the line AB.

Check your answers at the end of this unit

In the worksheet outlined above the work is restricted to lines with positive gradient, zero gradient (horizontal lines, with equations of the form $y = k$, where k is a real number) or no gradient (vertical lines, with equations of the form $x = p$, where p is a real number)

The next step is to introduce lines with negative gradient.

The following diagram could be used.



For one step across you have to move down to reach the line again. Lines with negative slope make an obtuse angle with the positive x-axis. Lines with positive gradients make acute angles with the positive x-axis.

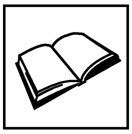
Unit 2, Practice activity 1

1. The suggestions in worksheet W4 need to be fine tuned to different attainment levels of pupils in the class. Describe in detail how you would do that.
2. Write a worksheet for pupils of different attainment levels to develop the concept of negative gradients. Justify the structure of your worksheet and give the expected working.
3. In self mark exercise 1 question 6 you derived a general expression for the gradient of a line through two points with given coordinates. Would you want your pupils to be able to recall and apply the ‘formula’ or would you want them to refer to a diagram when finding gradients? Justify your answer.

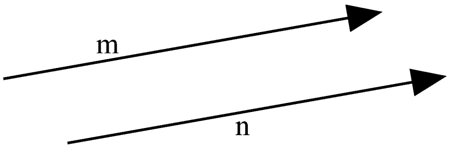
   How important, in general, is it for pupils to be able to recall ‘formula’ and how important is it to relate concepts to a ‘picture’ or ‘diagram’?

Present your assignment to your supervisor or study group for discussion.

Section A2: Parallel lines

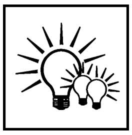


If the angle between two lines in a plane is 0° the lines are called parallel. The (perpendicular) distance between the two lines is constant (and could be zero).



The lines m and n are parallel to each other. Notation used is m || n. In the diagram arrows are placed to indicate parallel lines.

The following is an outline of a worksheet that can be set to pupils to discover the relation between parallel lines and their gradients. Pupils are expected to work in groups and discuss their work with each other.



Worksheet 5 (outline)

Objective: Pupils should discover that (i) parallel lines have the same gradient (ii) if two lines have the same gradient the lines are parallel.

1. Draw the lines with the following equations using the same coordinate grid

   Remember to follow the four steps
   (i) find and tabulate the coordinates of three points on the line
   (ii) plot the points on a labelled coordinate grid. Use X (two small crossing lines) to indicate the position of the point.
   (iii) join the points, extending the line on both sides
   (iv) label the line with its equation

   $y = x$

   $y = x + 1$

   $y = x + 2$

   $y = x - 1$

   $y = x - 2$

   a) What are the gradients of the lines?

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

2. Draw and label the lines with the following equations on a coordinate grid

   $y = 2x$

   $y = 2x + 3$

   $y = 2x - 2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

3. Draw and label the lines with the following equations on a coordinate grid

   $y = \frac{1}{2}x$

$y = \frac{1}{2}x + 4$

$y = \frac{1}{2}x - 3$

Copy and complete:
The lines all have as a gradient ..... and are .... to each other.

4. Draw and label the lines with the following equations on a coordinate grid

   $y = {}^{-}2x$

   $y = {}^{-}2x + 1$

   $y = {}^{-}2x - 2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

5. Draw and label the lines with the following equations on a coordinate grid

   $y = 2$

   $y = 3$

   $y = {}^{-}2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

6. Draw and label the lines with the following equations on a coordinate grid

   $x = 2$

   $x = 3$

   $x = {}^{-}2$

   Copy and complete:
   The lines all have as a gradient ..... and are .... to each other.

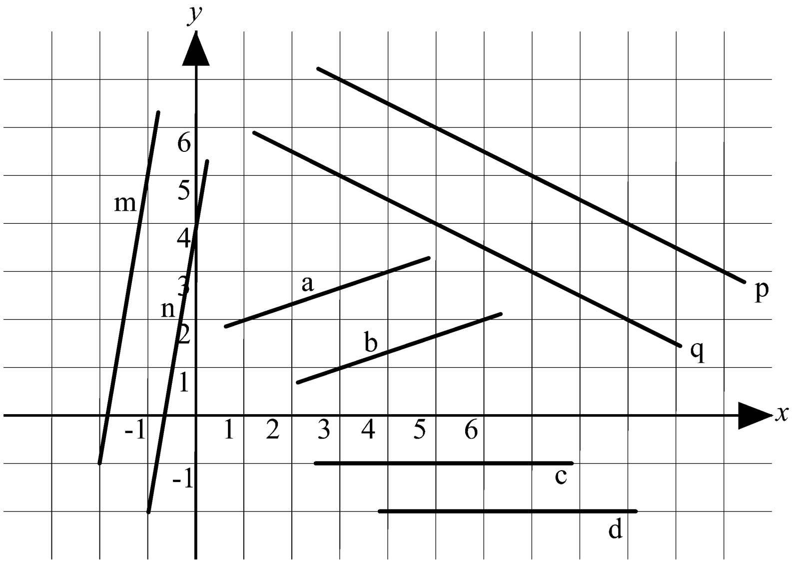
   Conclusion from question **1 - 6**:
   If two lines have an equal gradient then ....

7. Look at each set of three equations of lines.
   Which pair of lines in each set of three are parallel to each other?

a) $y = 4x - 7$	$y = 3x - 1$	$y = 4x$
b) $y = 2x + 3$	$y = 4 + 2x$	$y = 2 + 3x$
c) $y = 1 + 3x$	$y = 3 + x$	$y = 3$
d) $y = 2 + x$	$y = 2$	$y = {}^{-}2$

   e) What is the gradient of the parallel lines in a - d?

8. The diagram illustrates pairs of lines that are parallel to each other.



a) find the gradients of the lines a and b

b) find the gradients of the lines m and n

c) find the gradients of the lines p and q

d) find the gradients of the lines c and d

What can you say about the gradients if two lines are parallel?

Complete the statement: If two lines are parallel then ....

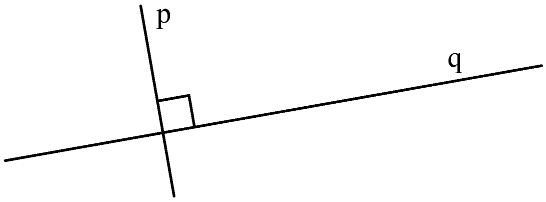
Self mark exercise 2

1. Work the questions in worksheet 5.
2. What concept is developed in questions 1 - 7 of worksheet 5?
3. What concept is developed in question 8 of worksheet 5?
4. Has an inductive or deductive method been used in worksheet 5? Explain.

Check your answers at the end of this unit

Section A3: Perpendicular lines

If the angle between two lines in a plane is 90° the lines are said to be perpendicular to each other.



The lines p and q are perpendicular to each other. The notation used is $p \perp q$. In a diagram a small square appears in the right-angle.

For parallel lines two relationships were considered
(i) If $m \parallel n$ then $grad_m = grad_n$ ($grad_n$ means "gradient of n")
(ii) If $grad_m = grad_n$ then $m \parallel n$

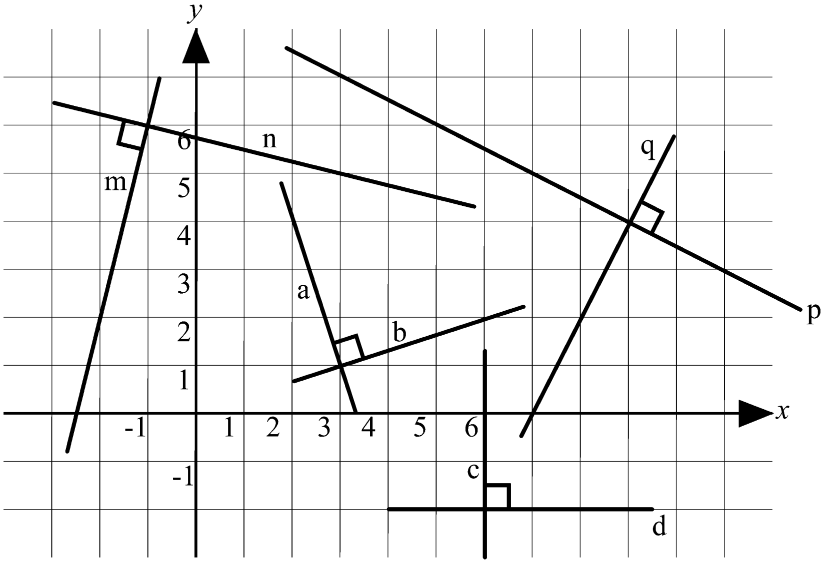
For perpendicular lines similarly two relationships are to be developed
(i) If $p \perp q$ then $grad_p \times grad_q = {}^-1$ (provided neither p or q is a horizontal or vertical line)
(ii) If $grad_p \times grad_q = {}^-1$ then $p \perp q$.

Below you find guided questions for pupils to discover the above relationships. Pupils are expected to work in groups and discuss their work with each other.

Worksheet 6 (outline)

Objective: Pupils to discover (i) if lines are perpendicular (and not parallel to the axes) the product of their gradients is $^-1$ (ii) if the product of the gradients of two lines is $^-1$ then the lines are perpendicular.

1. The diagram illustrates pairs of lines that are perpendicular to each other.



a) Find the gradients of the lines a and b, and the product of the gradients.

b) Find the gradients of the lines m and n, and the product of the gradients.
c) Find the gradients of the lines p and q, and the product of the gradients.
d) Find the gradients of the lines c and d, and the product of the gradients.
e) What is special about the situation in d?
f) What can you say about the product of the gradients if two lines are perpendicular?

Complete the statement:
If two lines are perpendicular to each other then ...... provided ......

2. Draw and label the lines with the following equations on a coordinate grid
$y = 2$
$x = 3$

   a) What are the gradients of the lines? What is the product of their gradients?
   b) What is the angle the two lines make with each other at their point of intersection?

   Copy and complete:
   The two lines are ...... to each other, however, the product of their gradients is not ......

3. Draw the lines with the following equations using the same coordinate grid

$$y = x + 1 \text{ and } y = {}^{-}x + 3$$

   a) What are the gradients of the lines? What is the product of their gradients?
   b) What is the angle the two lines make with each other at their point of intersection?

   Copy and complete:
   The product of the gradients of the lines is ...... and the lines are ...... to each other.

4. Draw and label the lines with the following equations on a coordinate grid

$$y = {}^{-}2x + 3 \qquad y = \frac{1}{2}x - 2$$

   a) What are the gradients of the lines? What is the product of their gradients?
   b) What is the angle the two lines make with each other at their point of intersection?

   Copy and complete:
   The product of the gradients of the lines is ...... and the lines are ...... to each other.

5. Draw and label the lines with the following equations on a coordinate grid

$$y = {}^{-}3x + 6 \qquad y = \frac{1}{3}x - 4$$

   a) What are the gradients of the lines? What is the product of their gradients?
   b) What is the angle the two lines make with each other at their point of intersection?

Copy and complete:
The product of the gradients of the lines is ...... and the lines are ...... to each other.

6. Look at each set of three equations of lines.
   Which pair of lines in each set of three are perpendicular to each other?

   a) $y = 4x - 7$ $\quad y = \frac{1}{4}x - 1$ $\quad y = {}^-4x$

   b) $y = 2x + \frac{1}{2}$ $\quad y = 4 - 2x$ $\quad y = 2 - \frac{1}{2}x$

   c) $y = 3 - xy = {}^-3 + x$ $\quad y = \frac{1}{3}$

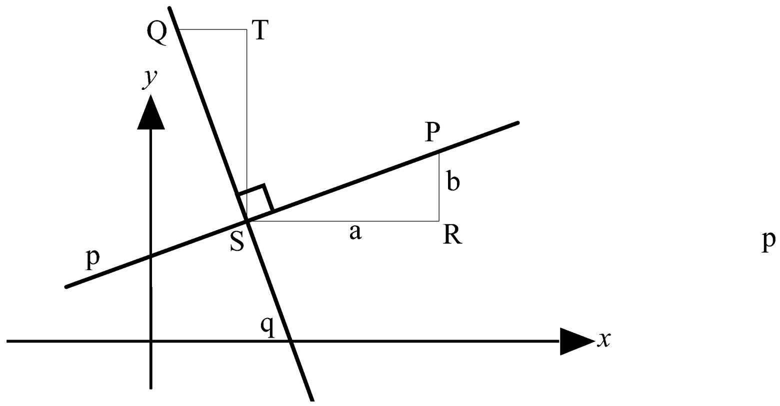
   d) $y = {}^-2$ $\quad y = \frac{1}{2}$ $\quad x = {}^-4$

Self mark exercise 3

1. Work the questions in worksheet 6.
2. What concept is developed in questions 1 - 2 of worksheet 6?
3. What concept is developed in questions 3 - 6 of worksheet 6?
4. In worksheet 6 an inductive method is used. As a teacher you should be able to give a deductive proof as a few high attainers might appreciate a deductive proof.

   a) To prove:

   If $p \perp q$ then $\text{grad}_p \times \text{grad}_q = {}^-1$ (provided neither p or q is a horizontal or vertical line).
   Try to prove, deductively, the statement. If you get stuck, read the following hints.
   (i) Draw a diagram with two lines p and q perpendicular to each other.



   If S, the point of intersection of p and q has coordinates (x, y) and P is a point on p with coordinates $(x + a, y + b)$, find the gradient of the line p using the right-angled triangle SRP.
   As q is perpendicular to p, q is obtained by rotating p through 90°. The triangle SRP moves in this rotation to position STQ. Use the triangle (you know the lengths of the sides) to find the gradient of line q.

   Find the product of the two gradients.
   (ii) Explain the part "provided neither p or q is a horizontal or vertical line".

   b) To prove:

   If $\text{grad}_p \times \text{grad}_q = {}^-1$ then $p \perp q$.
   Try to prove, deductively, the statement. If you get stuck, read the following hints.

   (i) Let the gradient of line p be m, then the gradient of q will be $-\frac{1}{m}$ as the product is given to be ${}^-1$. Take the origin at the point of intersection of p and q. Write down the equation of p and the equation of q.

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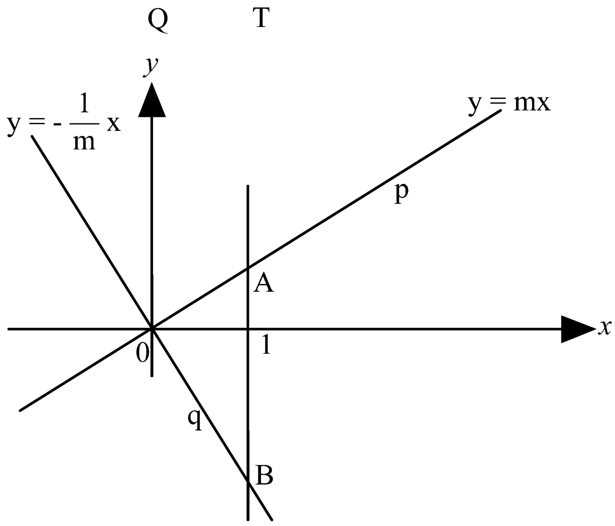


Find the coordinates of A, on p, if $x = 1$, and of B, on q, with $x = 1$.
Now consider triangle OAB. It is to be shown that angle AOB = 90°. Do this by showing that in the triangle the Pythagorean theorem holds.

Find $|OA|^2$, $|OB|^2$ and $|AB|^2$ and show that $|OA|^2 + |OB|^2 = |AB|^2$ and draw your conclusion.

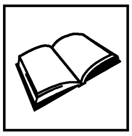
Check your answers at the end of this unit.

Unit 2, Practice activity 2

1. Critically evaluate the suggestions for worksheets W5 (parallel lines) and W6 (perpendicular lines). Make adaptations appropriate for your class and prepare a lesson plan for covering the relationships as covered in worksheets 5 and 6. Carry out the lessons and write an evaluative report. Do not forget to include the lesson plan and any other material developed for the lesson.

Present your assignment to your supervisor or study group for discussion.

Section B: The equation of a straight line



Plotting corresponding values of x and y satisfying a **linear equation** of the form $y = mx + n$ will give a graph of a straight line. It includes all straight lines except vertical lines. Vertical lines have no gradient and have equations of the format $x = k$, where k is an real number.

A format covering ALL lines is $ax + by = c$, which is the **general** equation of a straight line.

The format $y = mx + n$ is frequently used as 'general' equation. This is fine when one is sure that no vertical line is involved.

Section B1: Exploring the meaning of *b* in *y* = *ax* + *b*

In section A1 you looked at a guided activity for pupils to identity the gradient with the coefficient of x in $y = ax + b$. The following outline of a worksheet is to guide pupils to interpreting the value of b in the equation $y = ax + b$ as the y-intercept.

Worksheet 7 (outline)

Objective: Pupils to discover that in the equation of a line $y = ax + b$, the value of b is the y-intercept (directed distance from O to the point of intersection of the line and the y-axis).

1. a) On the same coordinate grid draw and label the lines with equations
   $y = 2x - 4$
   $y = 2x - 2$
   $y = 2x$
   $y = 2x + 3$

   b) Copy and complete these sentences:
   The line with equation $y = 2x - 4$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x - 2$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x$ crosses the y-axis at the point with y coordinate ....
   The line with equation $y = 2x + 3$ crosses the y-axis at the point with y coordinate ....

   c) Where will the line with equation $y = 2x + 9$ cross the y-axis?

   d) Where will the line with equation $y = 2x - 56$ cross the y-axis?

   e) Where will the line with equation $y = 2x + p$ cross the y-axis?

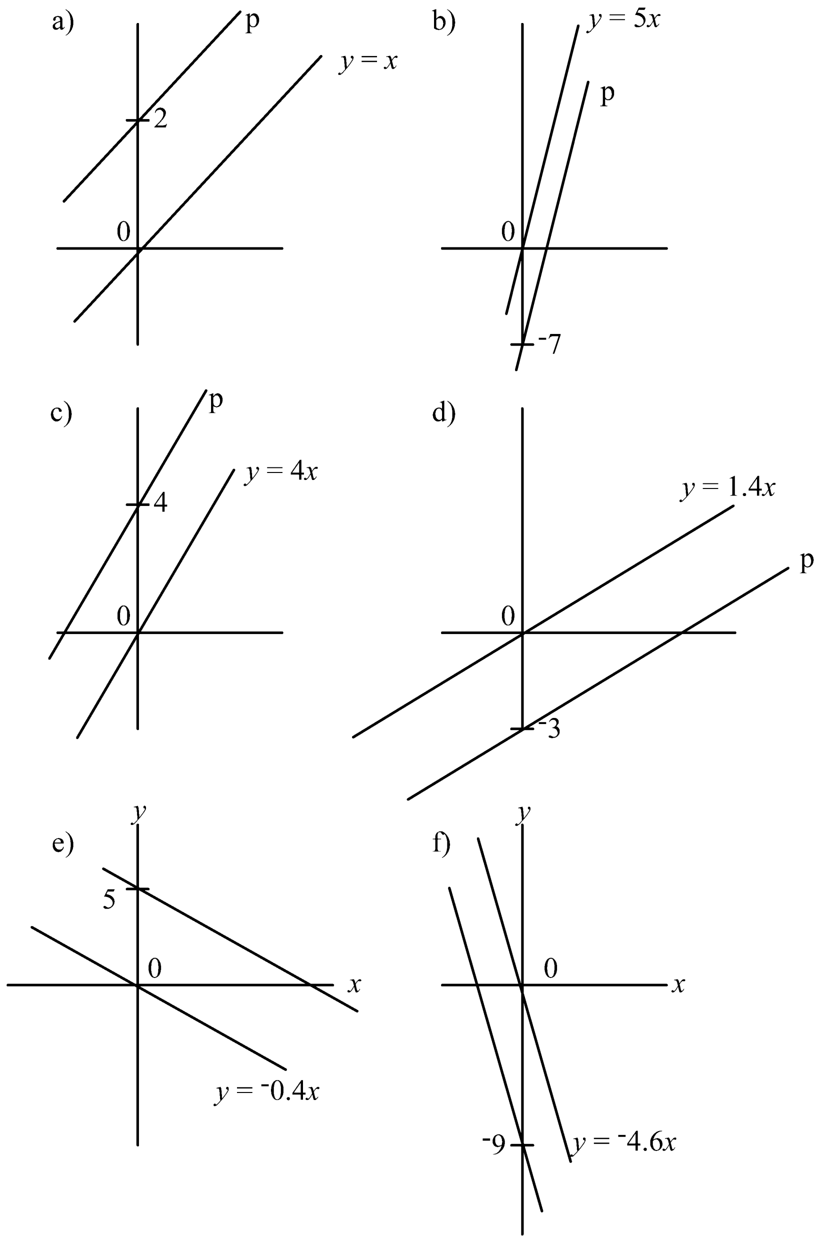
> The y coordinate of the point where the line with equation $y = ax + b$ crosses the y-axis is called the ***y*-intercept** of the line.

2. What is the y-intercept of the lines with equation

   a) $y = 7x - 9$ b) $y = 7 + 9x$ c) $y = x + 16$ d) $y = {}^{-}8 + 3x$

   e) $y = {}^{-}3x + 5$ f) $y = -\frac{1}{3}x - 5\frac{1}{4}$ g) $y = {}^{-}4.3x + 5.6$

3. What is the equation of the line p in each case? Line p is parallel to the line through O.



In the equation of the line $y = ax + b$:

a is the gradient or step size and b is the y-intercept.

The format of an equation of a line in word formula:

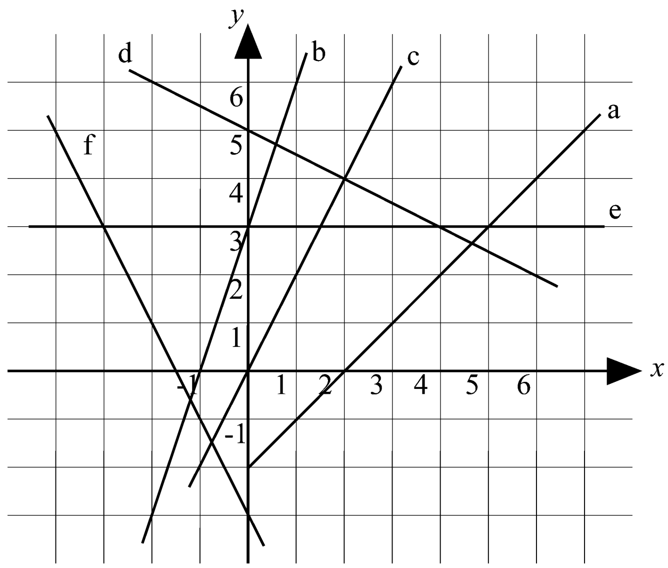
y = (gradient) x + (y-intercept)

4. For each line a, b, c, d, e and f

   a) What is the y-intercept of the line?

   b) What is the gradient of the line?

   c) What is the equation of the line?



Section B2: Plotting straight line graphs, given the equation of the line

In the plotting of straight line graphs, given their equation, the y-intercept and x-intercept are frequently included as they can be computed fairly easily (putting $x = 0$, or $y = 0$ respectively).

Pupils should be given sufficient practice in plotting graphs.

To plot the graph of the line with equation $y = ax + b$, you need the coordinates of three points (in theory, two are sufficient, but students make fewer errors if they choose three points).

Choose the following three points (if possible):

(i) the point where the line crosses the y-axis

(ii) the point where the line crosses x-axis

(iii) any other point

The x coordinate of the point where the line crosses the x-axis is called the x intercept of the line.

For example, to plot the graph of the line with equation $y = 3x - 6$:

$x = 0 \Rightarrow y = {}^{-}6$ (the y intercept)

$y = 0 \Rightarrow 0 = 3x - 6 \Rightarrow x = 2$ (the x intercept)

$x = 4 \Rightarrow y = 3 \times 4 - 6 = 12 - 6 = 6$.

Tabulated:

x	0	2	4
y	$^{-}6$	0	6

Section B3: Drawing lines with equations *px* + *qy* = *r*

The relationship between the y coordinate and x coordinate can be given in a form different from $y = ax + b$.

To draw the graph of lines with equations written in a different format, for example as

$x + y = 4$ or $2x + y = 7$ you follow the normal four steps:

(i) find and tabulate the coordinates of three points on the line: the point where the line crosses the y-axis, where it crosses the x-axis and one other point

(ii) plot the points on a labelled coordinate grid. Use x (two small crossing lines) to indicate the position of the point

(iii) join the points, extending the line at both sides

(iv) label the line with its equation

Self mark exercise 4

1. Answer the questions in worksheet 7.
2. Using the same coordinate grid draw and label the lines with the following equations. Follow the steps outlined above.

   a) $y = 2x - 5$ b) $y = 4 - 4x$ c) $y = x + 2$ d) $y = \frac{-1}{2}x + 3$

3. Using the same coordinate grid draw and label the graphs of the lines with these equations:

   a) $x + y = 6$ b) $x + \frac{1}{2}y = {}^{-}4$ c) ${}^{-}2x + 3y = 6$ d) $3x - 4y = 12$

   e) $2x + y = 0$ f) $x - y = {}^{-}5$

Check your answers at the end of this unit

Section B4: Finding equations of lines through two given points

The relationship between the y-coordinate and x-coordinate for non vertical lines can be given in the form $y = ax + b$ or in word formula y = (gradient/step size)x + (y-intercept). A line is fixed if you know two points on the line as there is only one line through two points. To find the equation of the line given two points, find the values of a and b in the relation $y = ax + b$.

Example

Find the equation of the line through the points with coordinates P (1, 2) and Q(3, 10).

Step 1: find the gradient of the line

To move from P to Q: 2 steps to the right and 8 steps up.

If you move 1 step to the right you will move 4 steps up (divided by 2)

The gradient or step size is 4.

Write the equation as you know it now: $y = 4x + b$. (I)

Step 2: find the value of b (the y-intercept)

Substitute the coordinates of either P or Q in (I) and solve the equation for b.

Substituting P(1, 2) gives: $2 = 4 \times 1 + b$

$2 = 4 + b$

Use the cover up method to find b: "What to add to 4 to get 2?"

Answer: $^{-}2$.

So $b = {}^{-}2$

The equation of the line is therefore $y = 4x - 2$. (II)

Check by substituting in (II) the x-coordinates of the other point Q(3, 10) and checking whether the correct y-coordinate is obtained.

If $x = 3$ $\quad y = 4 \times 3 - 2 = 12 - 2 = 10$ which is indeed the y-coordinate of the given point Q.

Self mark exercise 5

1. Find the equation of the lines through the points A and B if

   (i) A(1, 2) and B(3, 6)
   (ii) A(3, 1) and B($^-$5, 7)
   (iii) A($^-$2, 2) and B(0, 10)
   (iv) A($^-$5, 2) and B($^-$1, $^-$6)
   (v) A($^-$2, 2) and B($^-$2, 10)
   (vi) A(3, 1) and B($^-$7, 1)

2. Find the equation of the line through the points A and B if A(x_1, y_1) and B(x_2, y_2).
3. What is (i) the gradient (ii) y-intercept (iii) x-intercept of the line with equation $ax + by = c$, provided $a \neq 0$ and $b \neq 0$ at the same time?
4. Explain what happens if (i) $b = 0$ (ii) $a = 0$ in question 3.

Check your answers at the end of this unit.

Unit 2, Practice activity 3

1. Write a detailed lesson plan and outline for covering with your pupils the finding of equations of straight lines. Ensure you take into account the different attainment levels in your class.
2. Write 5 challenging questions related to finding equations of straight lines.
3. The coordinates of the vertices of triangle ABC are A(0,0), B(6, 3) and C(1, 9).
   a) Find the coordinates of the midpoint D of BC and the equation of the line AD (the median).
   b) Find the coordinates of the midpoint F of AB and the equation of the median CF.
   c) Find the equation of the median BE.
   d) Show that the three medians pass through one point and find the coordinates of this point Z (the centroid of the triangle).
   e) Find the ratio in which the median divide each other, i.e., the ratios AZ :ZD, BZ : ZE and CZ : ZF.
   f) Find the centroid of a triangle with coordinates of the vertices ($x1$, $y1$), ($x2$, $y2$) and ($x3$, $y3$).
4. The coordinates of the vertices of triangle ABC are A(0, 0) B(6, $^-$8) and C(6, 3).
   a) Find the equations of the three altitudes (line from the vertex perpendicular to the opposite side) AD, BE and CF.
   b) Show that the three altitudes are concurrent (pass through one point) and find the coordinates of this point.

Present your assignment to your supervisor or study group for discussion.

Summary

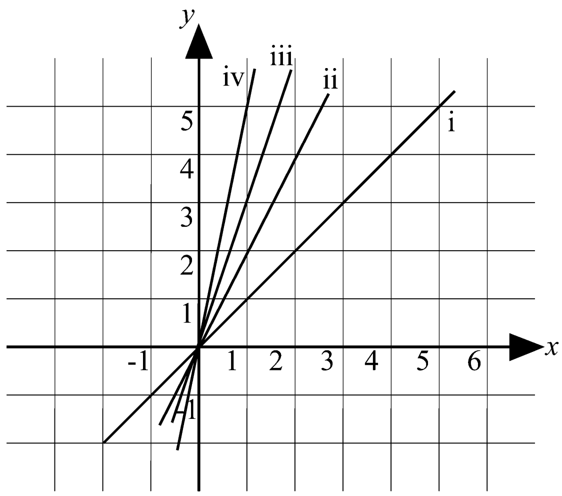
This unit continues the practice of having students learn geometrical properties, such as parallelism, through induction and thorough Cartesian rather than Euclidian concepts. Still, many students at this age experience difficulty making the conceptual links between spatial and algebraic expressions for the same "shape"; the unit recommends lots of exercises!

Unit 2: Answers to self mark exercises

Self mark exercise 1

1. Worksheet 4

   1. a)



   b) $y = 4x$ c) $y = x$ d) coefficient of x

   2. a) $y = 5x$ b) $y = 4x$ c) $y = 3\frac{3}{4}x$ d) n$y = 4.11x$

   3. a) $y = 3x$ b) $y = \frac{1}{2}x$ c) $y = 5x$

   4. a) 1, b) 3. c) 2, d) $\frac{1}{2}$

   5. a) 2, 4, 2

      b) (i) 3 (ii) 2 (iii) 3 (iv) $\frac{4}{3}$ (v) 1 (vi) 0 (vii) undefined

2. No. Vertical lines have the format $x = k$, where k is a constant. This format is not included in the form $y = ax + b$ which is a general equation for all line NOT parallel to the y-axis.
3. Horizontal, $y = p$, where p is a constant.
4. Vertical lines, $y = k$, where k is a constant
5. a) 0, $y = 0$ b) undefined, $x = 0$
6. See 7a

7. a) (i) $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-(y_2 - y_1)}{-(x_2 - x_1)} = \dfrac{y_1 - y_2}{x_1 - x_2}$

   (ii) division by zero is undefined

   b) tangent c) $\tan\theta = \dfrac{x}{y}$ = gradient

   d) if the angle increases from 0° to 90° the gradient increases from 0 to infinity.

   if the angle increases from 90° to 180° the gradient increases from negative infinity to 0

Self mark exercise 2

1. Worksheet 5

   1. gradient 1, the lines are parallel
   2. gradient 2, the lines are parallel
   3. gradient $\frac{1}{2}$, the lines are parallel
   4. gradient $^{-}2$, the lines are parallel
   5. gradient 0, the lines are parallel
   6. undefined gradient (or gradient ∞)
   7. (a) $y = 4x - 7$ and $y = 4x$ (b) $y = 2x + 3$ and $y = 4 + 2x$

      (c) none (d) $y = 2$ and $y = {^{-}2}$ (e) 4, 2, no parallel lines, 0
   8. (a) $\frac{1}{3}$ (b) 3 (c) $-\frac{1}{2}$ (d) 0

   If two lines are parallel lines then they have equal gradients.

2. If lines have equal gradient they are parallel.
3. If lines are parallel they have equal gradients.
4. inductive: conjectures were made from a number of specific examples

Self mark exercise 3

1. Worksheet 6

   1. a) $^{-}3$, $\frac{1}{3}$. Product $^{-}1$ b) 4, $-\frac{1}{4}$. Product $^{-}1$

      c) $-\frac{1}{2}$, 2. Product $^{-}1$ d) undefined, 0. Product undefined

      e) lines are parallel to the coordinate axes/horizontal, vertical line

      f) the product of their gradients is $^{-}1$, provided lines are not horizontal/ vertical
   2. a) 0, undefined; product undefined

      b) 90°, lines are perpendicular but product of their gradient is not $^{-}1$
   3. a) $^{-}1$, 1; product $^{-}1$

b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

4. a) $^{-}2$, $\frac{1}{2}$; product gradients is $^{-}1$

   b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

5. a) $^{-}3$, $\frac{1}{3}$; product gradients is $^{-}1$

   b) 90°, product of gradients is $^{-}1 \Rightarrow$ the lines are perpendicular to each other

6. a) $y = {}^{-}\frac{1}{4}x - 1$ and $y = 4x - 7$ b) $y = 2x + \frac{1}{2}$ and $y = 2 - \frac{1}{2}x$

   c) $y = 3 - x$ and $y = {}^{-}3 + x$ d) $y = {}^{-}2$ and $x = {}^{-}4$; $y = \frac{1}{2}$ and $x = {}^{-}4$

2. If two lines are perpendicular to each other (but not horizontal/vertical) then the product of their gradients if $^{-}1$.

3. If the product of the gradients of two lines is $^{-}1$ then the two lines are perpendicular to each other.

4. a) See text, gradient p is $\frac{b}{a}$, gradient q is $^{-}\frac{a}{b}$; vertical lines have no gradients, yet they are perpendicular to lines that are horizontal (gradient 0). The product is however not $^{-}1$.

   b) $A(1, m)$ $B(1, {}^{-}\frac{1}{m})$

   $|OA|^2 = 1 + m^2$ $|OB|^2 = 1 + \frac{1}{m^2}$ $|AB|^2 = (m + \frac{1}{m})^2$

   $= m^2 + 2 + \frac{1}{m^2}$

   Hence $|OA|^2 + |OB|^2 = |AB|^2$

Self mark exercise 4

1. Worksheet 7

1. b) $^{-}4$, $^{-}2$, 0, 3

   c) 9 d) $^{-}56$ e) p

2. a) $^{-}9$ b) 7 c) 16

   d) $^{-}8$ e) 5 f) $^{-}5\frac{1}{4}$

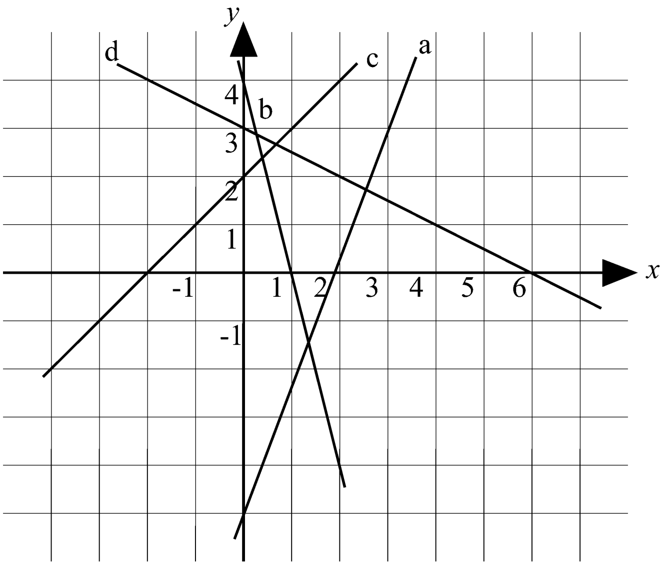
   g) 5.6

3. a) $y = x + 2$ b) $y = 5x - 7$ c) $y = 4x + 4$

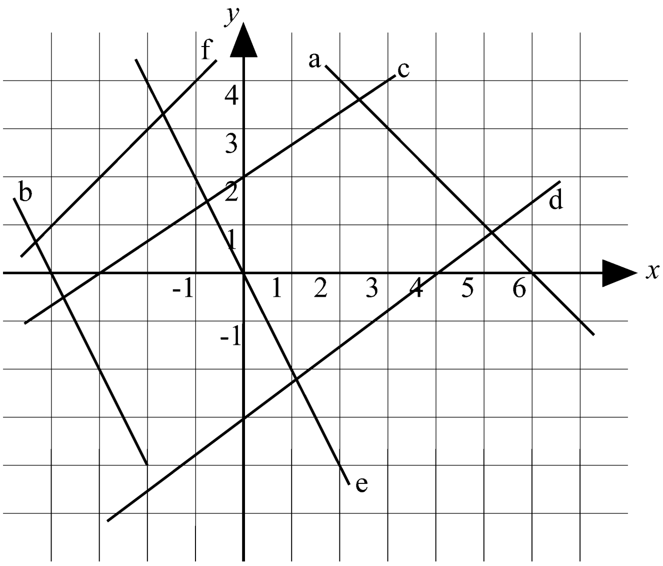
   d) $y = 1.4x - 3$ e) $y = 0.4x + 5$ f) $y = {}^{-}4.6x - 9$

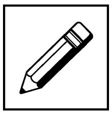
4. a) $^{-}2, 1, y = x - 2$ b) $2, 3, y = 3x + 2$ c) $0, 2, y = 2x$
d) $1, ^{-}2, y = x - 2$ e) $0, 3, y = 3$ f) $^{-}2, ^{-}3, y = 2x - 3$

2.



3.





Self mark exercise 5

1. (i) $y = 2x$ (ii) $y = {}^{-}\frac{3}{4}x + 3\frac{1}{4}$ (iii) $y = 4x + 10$

(iv) $y = {}^{-}2x - 8$ (v) $x = {}^{-}2$ (vi) $y = 1$

2. Step 1: gradient = $\dfrac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \dfrac{y_1 - y_2}{x_1 - x_2}$

$y = \dfrac{y_1 - y_2}{x_1 - x_2}\, x + b$ (i)

Step 2: Substitute (x_1, y_1) to find b. $y_1 = \frac{y_1 - y_2}{x_1 - x_2} x_1 + b$

$b = y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$

Placing the value of b in equation (i)

$y = \frac{y_1 - y_2}{x_1 - x_2} x + y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$

$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2} (x - x_1)$

3. (i) $-\frac{a}{b}$ (ii) $\frac{c}{b}$ (iii) $\frac{c}{a}$

4. If $b = 0$, the equation represents a vertical line. These lines have undefined gradient and no y-intercept.

   If $a = 0$, the equation represents a horizontal line. These lines have gradient 0 and no x-intercept.

Unit 3: Transformations I

Purpose of Unit 3

The purpose of this unit is to revise your knowledge on transformations: reflection, rotation, translation and enlargement. In the classroom situation reflection and rotation require a practical approach: folding and rotating tracing paper. A practical and intuitive approach of the linear transformations is used in this unit. The transformations can be done on plain or on coordinate grid relating the topic to coordinate geometry. The emphasis is on how you could present the topic to pupils in the class and the misconceptions of pupils to be aware of. Going through this unit might also help you to get your own understanding of reflection, rotation, translation and enlargement more clear.

Objectives

When you have completed this unit you should be able to:

- set activities at different levels of difficulty to pupils to draw reflections of given shapes in given mirror lines
- set activities at different levels of difficulty to pupils to draw the line of reflection given a shape and its image under a reflection
- find the equation of the mirror line given an object and its image on a coordinate grid
- list four properties of reflection, i.e.,

  In a reflection

  (i) the line segment connecting a point with its image is perpendicular to the mirror line (unless the point is on the mirror line)

  (ii) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

  (iii) the image of a line (segment) parallel to the mirror line is parallel to the mirror line

  (iv) the image m′ of a line m (not parallel to the mirror line) meets m on the mirror line
- develop a discovery activity for pupils to discover the above four properties of reflection
- state the common errors/misconceptions of pupils when reflecting a shape
- take appropriate remedial steps to overcome pupils’ misconceptions in reflection of shapes
- set activities at different levels of difficulty for pupils to draw the image of a shape under rotation (multiples of 90°) given the shape and its centre

- describe fully a rotation given the shape and its image under a rotation
- state common difficulties of pupils in rotating of shapes
- take appropriate remedial steps to overcome pupils' misconcepts in the rotation of shapes
- use column vectors to describe translations
- translate shapes given the shape and the translation vector
- state the different notations used to denote the translation vector
- use games to consolidate the concept of translation
- set a variety of activities at different levels for pupils to develop understanding of enlargements
- distinguish between shapes that are similar and shapes that are enlargements from each other

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

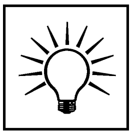
Section A: Reflection

Section A1: Reflection on reflection

Before you start on this unit reflect on your present practice in your classroom when covering the topic of 'reflection'. The following questions might guide you.

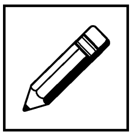
1. Write down an outline of the lessons in which the concept 'reflection of shapes' is covered. What is your starting point? What are the activities the pupils are involved in?
2. What do you cover and how do you do it? Do you cover for example: Reflections on square grid paper ? Reflections on plain paper? Reflections in horizontal, vertical, slanting lines? Reflections of shapes without reference to coordinates? Reflections of shapes and points with reference to the coordinates?
3. Write down all the properties of a reflection. Which of the properties do you want pupils to learn?
4. Write down the common errors made by pupils when reflecting shapes or finding lines of reflection. What do you do to prevent these common errors?

When going through this unit keep on returning to what you wrote down on your own practice.



Section A2: How to draw a reflection of a shape

What method do you use when having to obtain the reflection of a shape in a given mirror line? What method do you present to your pupils? Work the following exercise and reflect on HOW you are obtaining the image of each shape.



Self mark exercise 1

1. Copy the following shapes and find accurately their reflection in the mirror line (or line of reflection) m

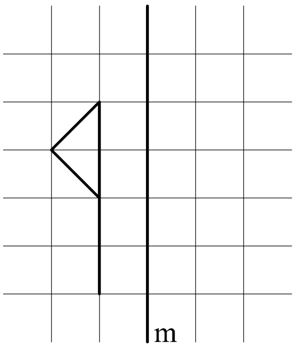


Figure 1

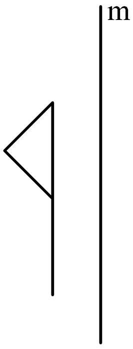


Figure 2

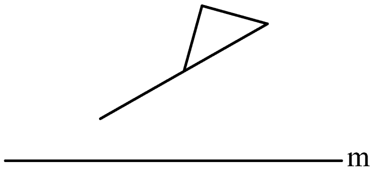


Figure 3

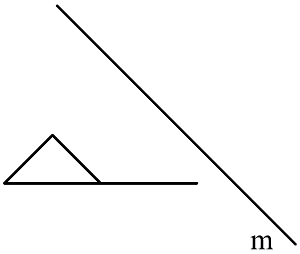


Figure 4

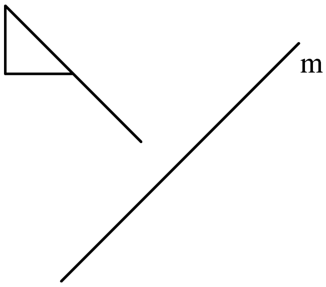


Figure 5

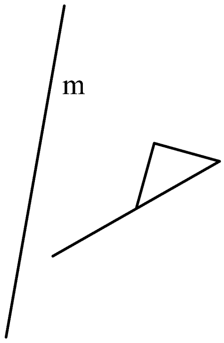
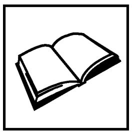


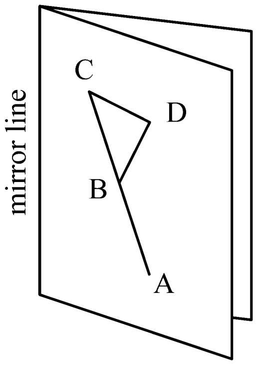
Figure 6

Check your answers at the end of this unit.



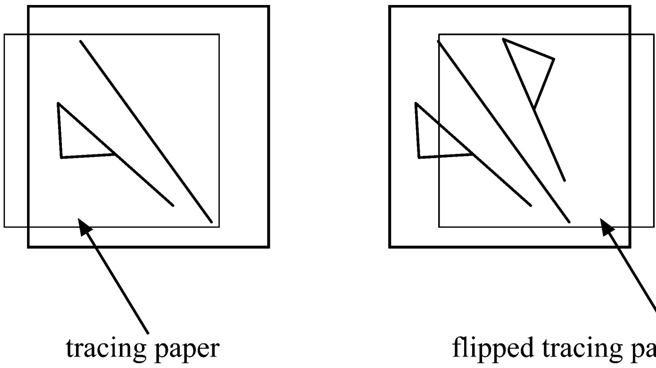
In the **initial stage** pupils are expected to ‘intuitively’ find the image of the object in the mirror line by

(i) folding the paper along the mirror line and using the point of a compass to mark the ‘corners’ of the reflected shape.



Piercing at A, B, C and D will give A′, B′, C′ and D′ at the other side of the mirror line. Joining these points will give the mirror image of the object.

(ii) using tracing paper. Place the tracing paper on top of the original and copy mirror line and object. Now flip the tracing paper and ensure the mirror line on the tracing is on top of the original mirror line. Piercing with compasses or sharp pencil will mark the position of the image.

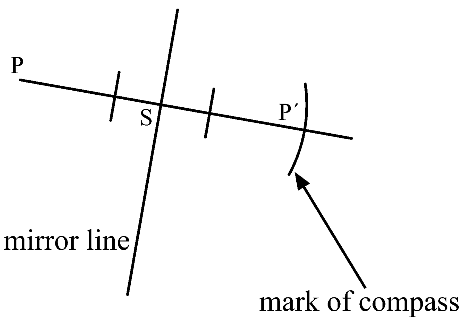


In a **later stage**, after pupils have discovered some of the properties of reflection, set square and ruler can be used.

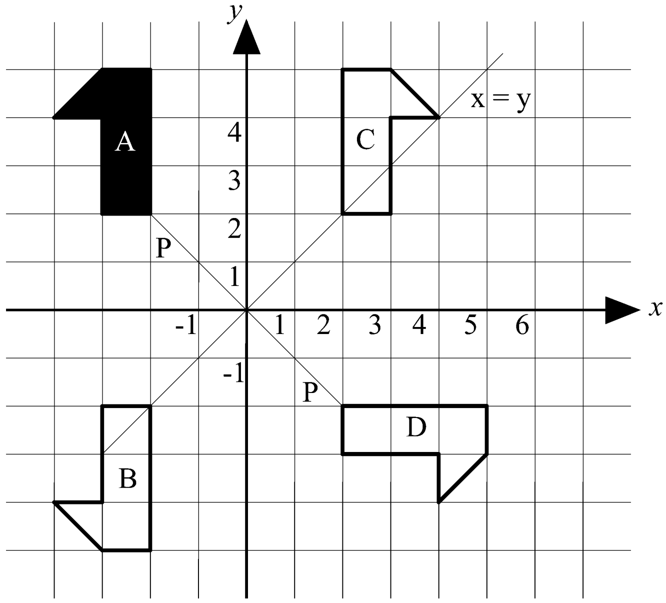
In turn the image of 'corner' points of the shape are found by

(i) drawing a line from the point P perpendicular to the mirror line (meeting the mirror line at S).

(ii) using a compass to ensure that PS = P′S, by placing point at S, making the opening equal to PS and circling at the other side to find P′. The process is repeated for all 'corner' points of the object.

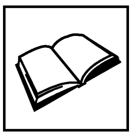


The following diagram is to remind you of reflection on a square grid.



The diagram illustrates reflection of shape A, the **object**, onto B, the **image**, by reflecting in the x-axis (the line with equation $y = 0$).
C is the image of A under a reflection in the y-axis (the line with equation $x = 0$).
D is the image of A under a reflection in the line with equation $x = y$.

To describe a reflection fully you are to give the equation of the **line of reflection** or **mirror line**.



Section A3: Pupils' difficulties with reflections

Difficulty level of questions on reflection of shapes in a given mirror line depends on

(i) ***Presence or absence of a grid.***
With grid lines given the facility increases.

(ii) ***The complexity of the shape to be reflected.***
When the shape consist of a single point the question of reflecting the point in a given mirror line is easier than when a more complicated shape is to reflected.

(iii) ***Slope of line of reflection/mirror line.***
Whether or not the mirror lines are horizontal/vertical or skew has an impact on the difficulty of the questions. In the latter case the questions become harder.

See the following illustration. Figure 1 is easier than the item in Figure 2, which is still easier than Figures 3 and 4.

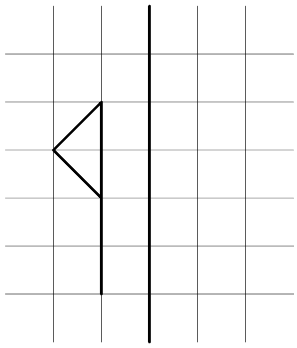


Figure 1

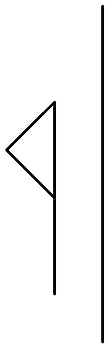


Figure 2

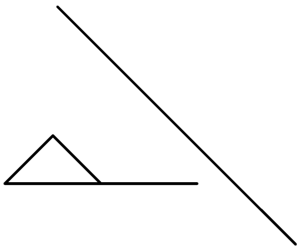


Figure 3

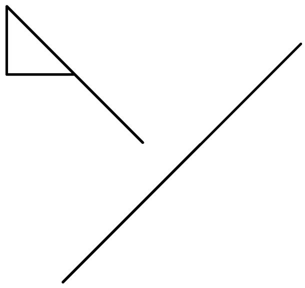


Figure 4

The same is true in cases where the object and the image is given, and the pupil is required to draw the mirror line, if any. (see Fig 5, 6, and 7).

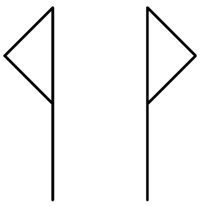


Figure 5

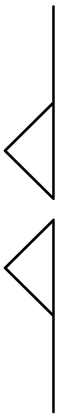


Figure 6

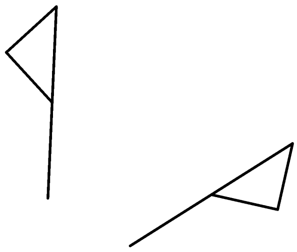


Figure 7

(iv) **Mirror line passes through the shape to be reflected or contains one of the line segments of the shape.**

Questions in which the mirror line has no common points with the shape to be reflected are easier for pupils to handle.

In the following figure, the situation in Figure 1 is easier than the situation in Figure 2.

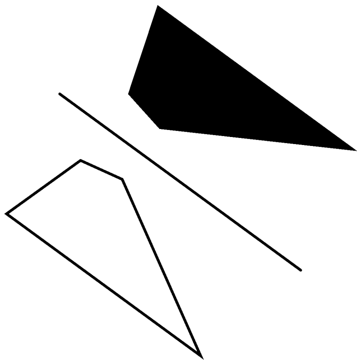


Figure 1

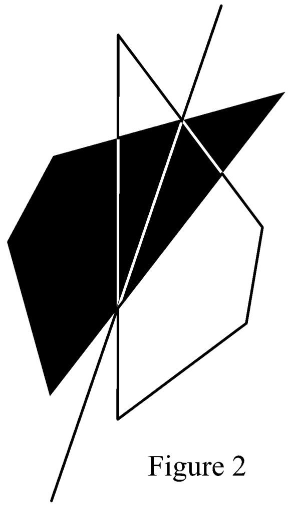
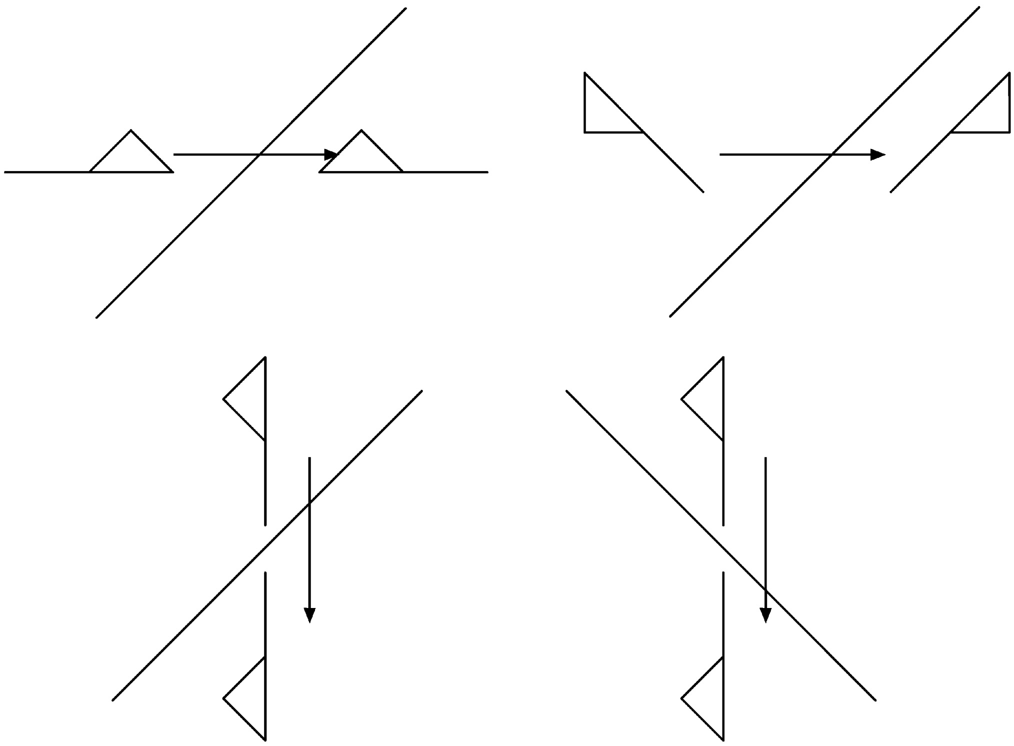


Figure 2

Section A4: Common errors of pupils

a) When the given mirror line is slanting the slope is ignored and the object is reflected horizontally or vertically. The presence or absence of a grid has no influence on this error.

Error workings are illustrated below.



b) If the object is horizontal or vertical the tendency to reflect horizontally (vertically) in a slanting mirror line is even stronger. In general there is a tendency to draw images parallel to the object.

Unit 3, Practice activity 1

1. Design a diagnostic test to find out whether pupils in your class have any of the misconceptions described on the previous page. Ensure all possible cases are covered: reflection with or without grid, line of reflection horizontal, vertical, slanting, position of the object with respect to the line of reflection. For each item state the objective: What error is the item to diagnose?
2. Administer the test and analyse the results.
3. A pupil makes the error described and illustrated above: the object is reflected horizontally or vertically with a line of reflection slanting.

   Describe in detail the four steps in remediation ((i) diagnosing the error by asking the pupil to explain what he/she did (ii) creating conflict in the mind of the pupil by using the pupil's method which clearly leads to a mental conflict situation (iii) setting activities to build the correct concepts and (iv) consolidation of the concept) you would take to help the pupil to overcome the error.

Present your assignment to your supervisor or study group for discussion.

Section A5: Objectives to be covered with pupils

Pupils should be able to

a) given the mirror line (line of reflection) and the shape, to find the image of the shape when reflected in the line (drawing or giving coordinates)
b) given the shape, the line of reflection and an 'image', to determine whether the given line is/is not a line of reflection
c) given a shape and an image of the shape, to find the mirror line (if any), i.e., draw the line and/or find its equation

Each objective is to be covered without a grid present and on a coordinate grid, with the three possible positions of the line of reflection: horizontal, vertical and slanting.

Properties of reflections that could be covered are:

(i) the line segment connecting a point with its image is perpendicular to the mirror line (unless the point is on the mirror line)

(ii) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

(iii) the image of a line (segment) parallel to the mirror line is parallel to the mirror line

(iv) the image m´ of a line m (not parallel to the mirror line) meets m on the mirror line

An investigative approach is to be used in covering these properties.

Section A6: Worksheets for pupils on reflection

On the next pages you will find suggestions for activities that can be presented to the pupils.

Worksheet 1 has as objectives (i) to practice/consolidate drawing of images of shapes given the shape and the mirror line (ii) to discover the following properties of a reflection:

a) the line segment connecting a point with its image is perpendicular to the mirror line

b) a point and its image are at the same distance from the mirror line (or the mirror line bisects the line segment joining a point and its image)

c) a point on the mirror is its own image

Worksheet 2 has as objectives (i) to practice/consolidate drawing of images of shapes given the shape and the mirror line (ii) to discover that a) the image m′ of a line m (not parallel to the mirror line) meets it on the mirror line b) the image of a line (segment) parallel to the mirror line is parallel to the mirror line.

Worksheet 3 has as objectives (i) to draw the line of reflection given a shape and its image (ii) to recognise whether or not a diagram represents a reflection.

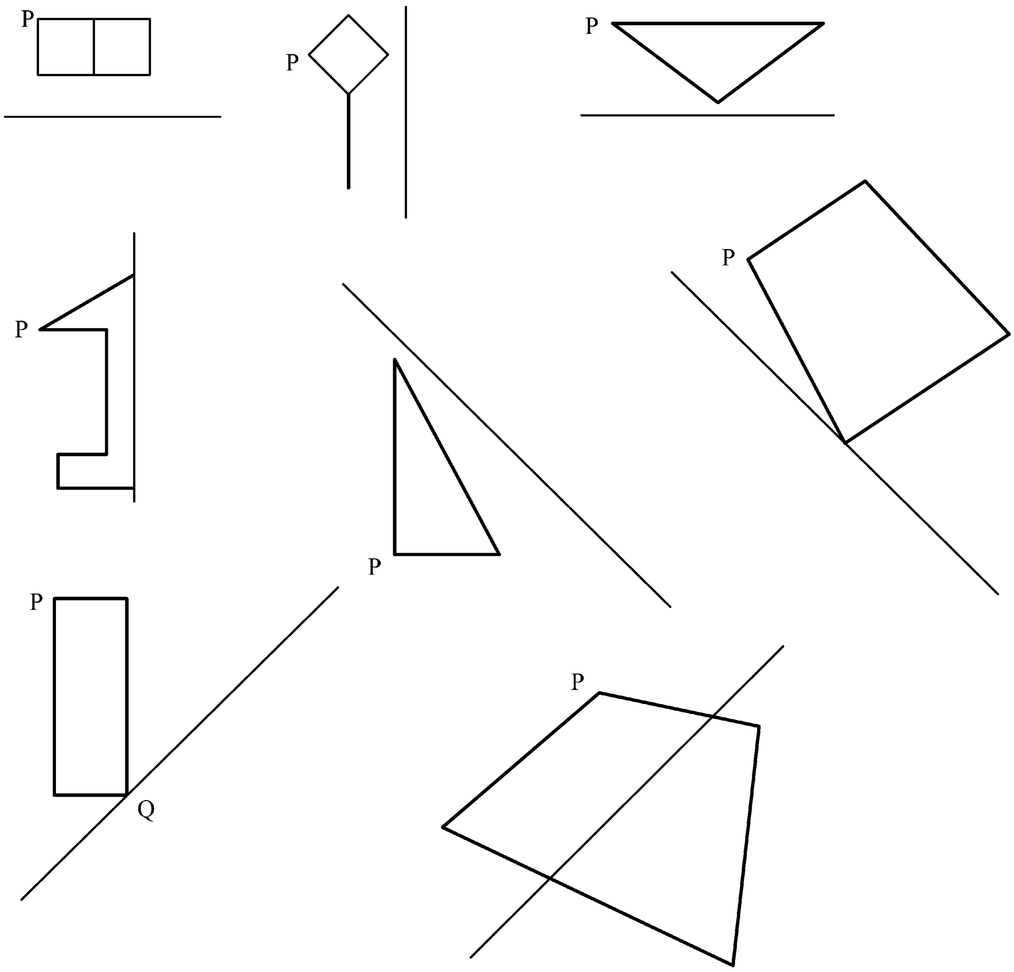
Worksheet 4 looks at reflection of points given the coordinates on a coordinate grid. Pupils are the discover relationships between the coordinates of the original point P(a, b) and the image of P when P is reflected in (i) the x-axis (ii) the y-axis (iii) the line with equation $x = y$ (iv) the line with equation $y = {}^{-}x$.

Worksheet 1

1. Copy the diagrams on a A4 paper, using tracing paper to trace the diagrams including the mirror line.

   Flip over your tracing paper and place mirror line of tracing on the mirror line of the original.

   Mark the positions of the image 'corners' and join them to show the position of the shape after it has been reflected. Mark the image of P, P′.

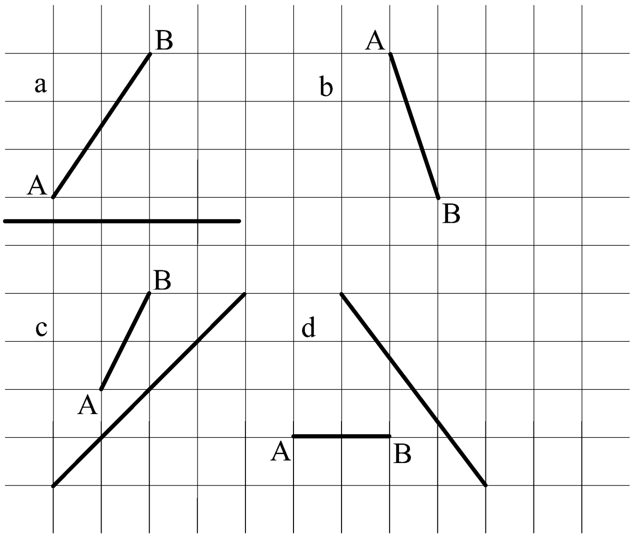


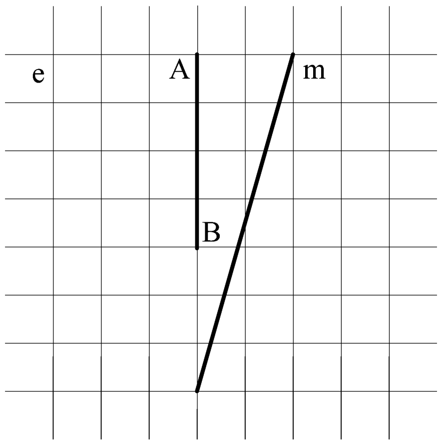
2. Draw in each diagram the line segment PP′.
3. In each case: What is the angle between the segment PP′ and the line of reflection?
   Complete the statement:
   In a reflection the line joining a point P with its P′ is ......
4. Mark in each diagram the point where the line PP′ meets the line of reflection S.
   Measure and compare the lengths of PS and P′S. What do you notice?

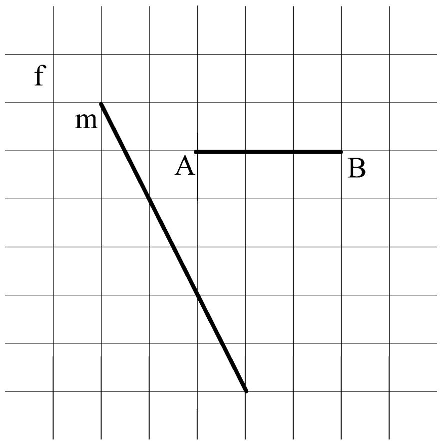
   Complete the statement:
   In a reflection the distance from P to the mirror line is ......
5. What is the image of the point Q? Make a statement about the image of points on the mirror line.

Worksheet 2

1. Copy the following diagrams on squared paper and draw the reflection of the line segments AB in the given mirror line. Label the image A´B´. Produce the line segment and its image until they meet.

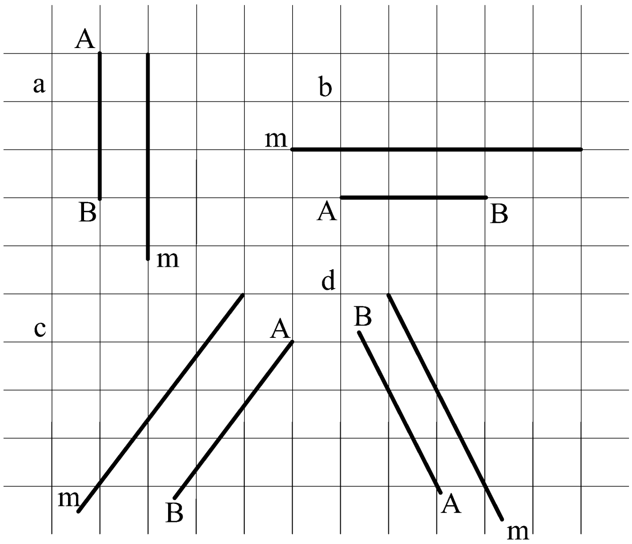






Compare the results in your group and complete the statement:
In a reflection a line and its image meet ......

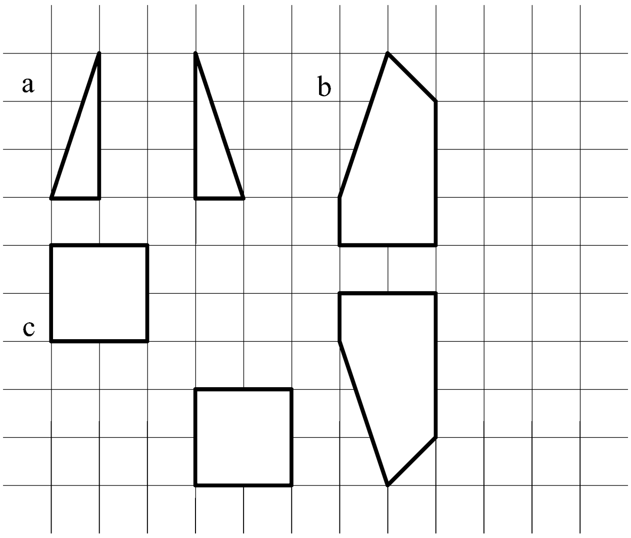
2. Check your statement for the following cases.
   Copy the diagram on squared paper and reflect AB. Find where AB and its image A´B´ meet.

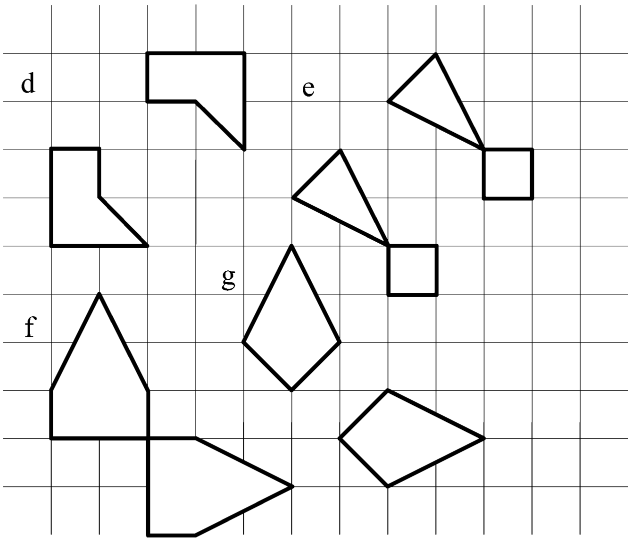


   Where do AB and A´B´ meet?
   Complete the statement:
   If a line is parallel to the mirror line the image of the line is ......

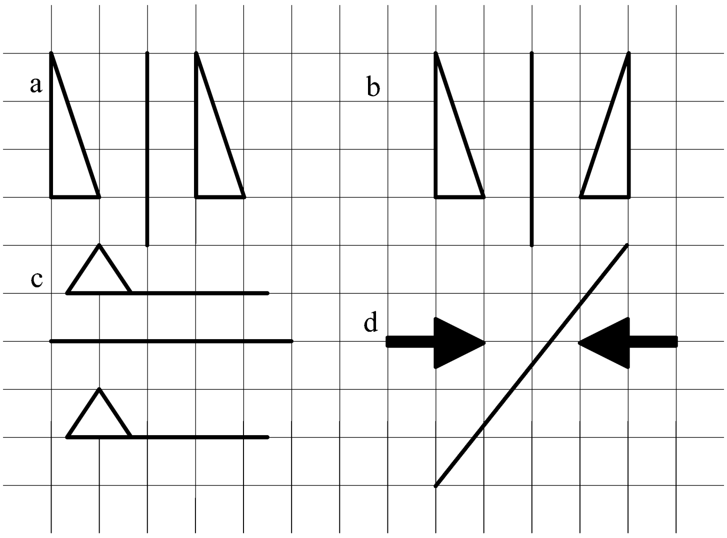
Worksheet 3

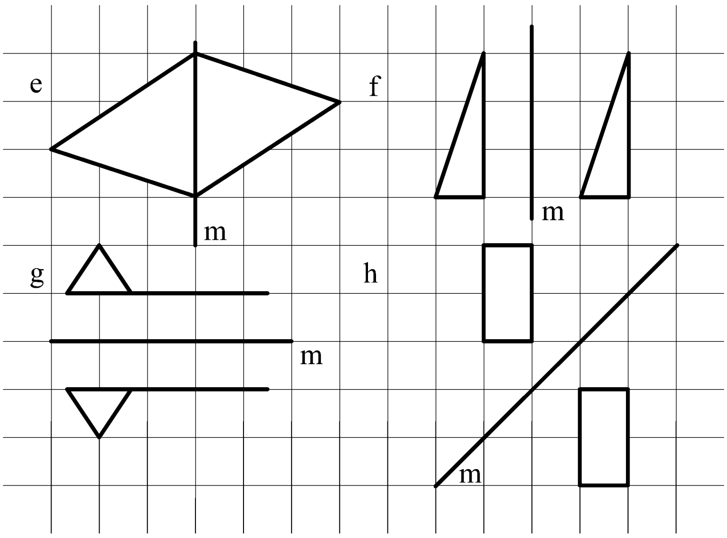
1. In each of the following diagrams one shape is the reflection of the other. Copy on squared paper and draw the position of the mirror line.





2. In which of the following diagrams is one shape the reflection of the other in the mirror line shown? Make a correct diagram for those that are not correct.





Worksheet 4

1. Use a coordinate grid to find the coordinates of the image of the following points P when reflected in the *y*-axis.

   Tabulate your results.

Coordinates original point	Coordinates of image
(3, 2)	
$(^-5, 2)$	
$(^-2, 3)$	
$(7, {}^-2)$	
$(^-4, {}^-5)$	
$(^-1, 3)$	
(4, 6)	
$(5, {}^-6)$	
$(^-3,b)$	
(a, 4)	
(a, b)	
(2a, 3b)	

2. Use a coordinate grid to find the coordinates of the image of the following points P when reflected in the *x*-axis.

   Tabulate your results.

Coordinates original point	Coordinates of image
(3, 2)	
$(^-5, 2)$	
$(^-2, 3)$	
$(7, {}^-2)$	
$(^-4, {}^-5)$	
$(^-1, 3)$	
(4, 6)	
$(5, {}^-6)$	
$(a, {}^-3)$	
(4, b)	
(a, b)	
(2a, 3b)	

3. Use a coordinate grid to find the coordinates of the image of the following points P when reflected in the line with equation $x = y$.

   Tabulate your results.

Coordinates original point	Coordinates of image
(3, 2)	
($^-5$, 2)	
($^-2$, 3)	
(7, $^-2$)	
($^-4$, $^-5$)	
($^-1$, 3)	
(4, 6)	
(5, $^-6$)	
($^-3$, b)	
(a, 4)	
(a, b)	
(2a, 3b)	

4. Use a coordinate grid to find the coordinates of the image of the following points P when reflected in the line with equation $y = {}^-x$.

   Tabulate your results.

Coordinates original point	Coordinates of image
(3, 2)	
($^-5$, 2)	
($^-2$,3)	
(7, $^-2$)	
($^-4$, $^-5$)	
($^-1$, 3)	
(4, 6)	
(5, $^-6$)	
($^-3$, b)	
(a, 4)	
(a, b)	
(2a, 3b)	

5. a) Look at your results in question 1 - 4. Summarise your findings by completing the following table.

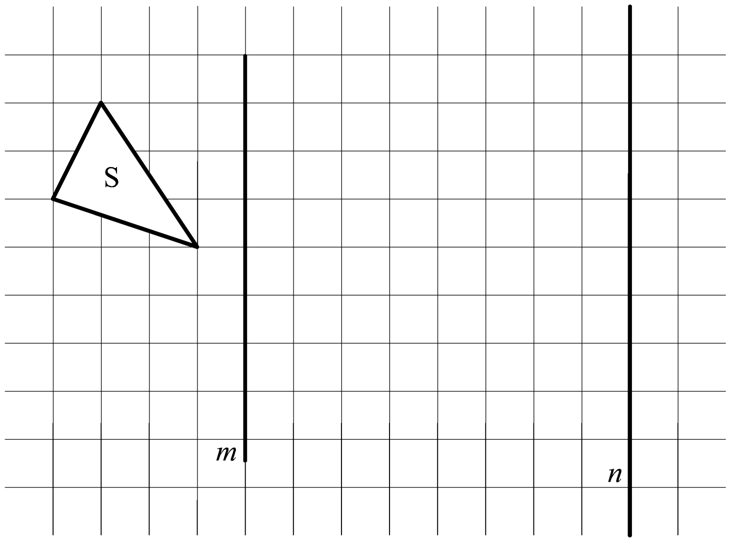
Coordinates original point	Equation of line of reflection	Coordinates of image
(a, b)	$x = 0$	
(c, d)	$y = 0$	
(e, f)	$x = y$	
(g, h)	$x = {}^-y$	

6. Complete the following table.

Coordinates original point	Equation of line of reflection	Coordinates of image
(3, $^-$2)	$x = 0$	
($^-$5, $^-$20)	$y = 0$	
($^-$2.3, 3.7)	$x = y$	
(9, $^-$2)	$x = {}^-y$	
	$x = 0$	($^-$5.6, 3.5)
	$y = 0$	(32, $^-$45)
	$x = y$	(12, $^-$8)
	$x = {}^-y$	($^-$4.7, 7.3)
(5, $^-$7)		($^-$7, 5)
(5, $^-$7)		($^-$5, $^-$7)
(5,$^-$ 7)		(5, 7)
(5, $^-$7)		(7, $^-$5)

Self mark exercise 2

1. Work the questions in worksheet 1.
2. Work the questions in worksheet 2.
3. Work the questions in worksheet 3.
4. Work the questions in worksheet 4.
5. Use squared paper.



Reflect the shape S in the mirror line *m* to give S′, reflect S′ in the mirror line *n* to give you S″. Investigate the relationship between S and S″ for different distances between lines *m* and *n*.

6. Make a table for several points, as in worksheet 4, to find the coordinates of P(a, b) when reflected in

   a) the line with equation $x = 3$

   b) the line with equation $y = 3$

   c) the line with equation $x + y = 3$

   d) the line with equation $x - y = 3$

7. The point P(x, y) is reflected in the line with equation $y = mx + n$; find the coordinates of P′ the image of P.

Check your answers at the end of this unit.

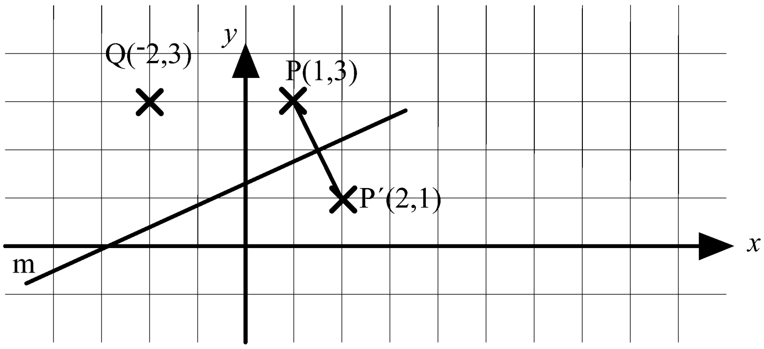
Section A7: Finding the equation of the line of reflection on a coordinate grid

If you are given a shape and its image under a reflection, you can find the line of reflection and its equation in most cases by careful inspection of the diagram.

The results you obtained in the previous exercises (the midpoint of the line segment joining point and its image is on the mirror line/line of reflection; the mirror line is perpendicular to the line joining a point and its image) allows also an algebraic approach.

Example

The line segment PQ with coordinates of the end points P(1, 3) and Q($^-2$, 3) is reflected. The image is P′Q′ where the coordinates of P′ are (2, 1). Find the equation of the line of reflection.



The coordinates of the midpoint of PP′ are $\left(\frac{1+2}{2}, \frac{3+1}{2}\right) = \left(1\frac{1}{2}, 2\right)$.

The gradient of PP′ is $\frac{3-1}{1-2} = {}^-2$.

If the line perpendicular to PP′ has a gradient p then $^-2 \times p = {}^-1$ and $p = \frac{1}{2}$.

The line of reflection passes through the midpoint of PP′ the point with coordinates $\left(1\frac{1}{2}, 2\right)$ and is perpendicular to PP′ with gradient $\frac{1}{2}$.

Hence the equation is of the format $y = \frac{1}{2}x + c$.

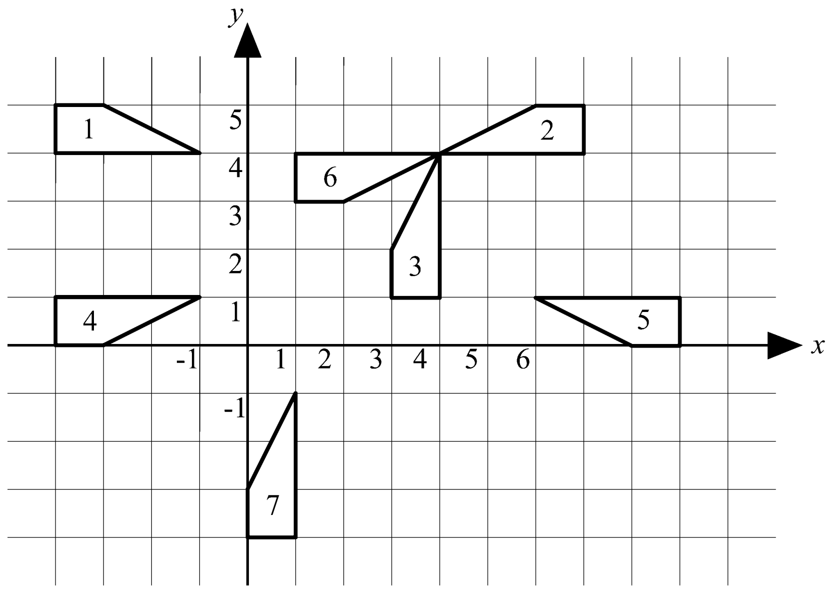
Substitution $\left(1\frac{1}{2}, 2\right)$ gives: $2 = \frac{1}{2} \times 1\frac{1}{2} + c$. So $c = 2 - \frac{3}{4} = 1\frac{1}{4}$.

The equation of the line of reflection is therefore $y = \frac{1}{2}x + 1\frac{1}{4}$.

N.B. The coordinates of Q are NOT relevant at all. It is sufficient to have given the coordinates of one point and its image under the reflection.

Self mark exercise 3

1. Describe fully the following reflections by giving the equation of the line of reflection.



a) 1 onto 2 b) 2 onto 3 c) 1 onto 4 d) 4 onto 5
e) 3 onto 6 f) 4 onto 7

2. Given are the coordinates of the vertices of the triangles:

$\Delta 1$: $(^-1, 2), (^-1, 5), (^-2, 5)$ $\Delta 2$: $(5, 2), (5, 5), (6, 5)$

$\Delta 3$: $(5, 0), (5, ^-3), (6, ^-3)$ $\Delta 4$: $(0, 3), (^-3, 3), (^-3, 4)$

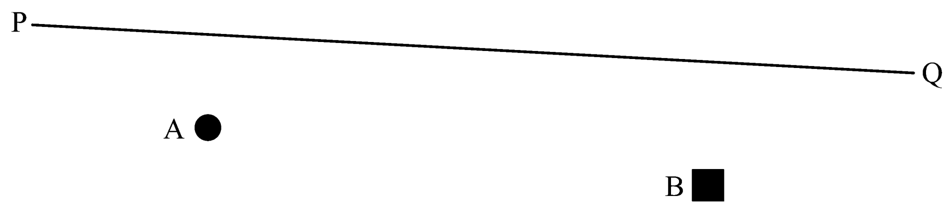
$\Delta 5$: $(^-1, 2), (2, 2), (2, 1)$

For the following mappings (i) draw the triangles on a square grid (ii) draw the mirror line (iii) find the equation of the line of reflection (mirror line)
a) $\Delta 1$ onto $\Delta 2$ b) $\Delta 2$ onto $\Delta 3$ c) $\Delta 1$ onto $\Delta 4$ d) $\Delta 1$ onto $\Delta 5$

Check your answers at the end of this unit.

Unit 3, Practice activity 2

1. Choose one or more of the activities on reflection presented above in the worksheets and self-mark exercises. Adapt the activity to meet the situation in your classroom and try them out. Write an evaluative report paying specific attention to the common errors diagnosed earlier.
2. Water utilities is to make a bore hole along the riverbed PQ.



The houses A and B are to be connected. Where is the bore hole to be made to minimise the length of pipe needed to connect A and B to the bore hole?

Present your assignment to your supervisor or study group for discussion.

Section B: Rotation

Section B1: Points to keep in mind as a teacher when covering rotation

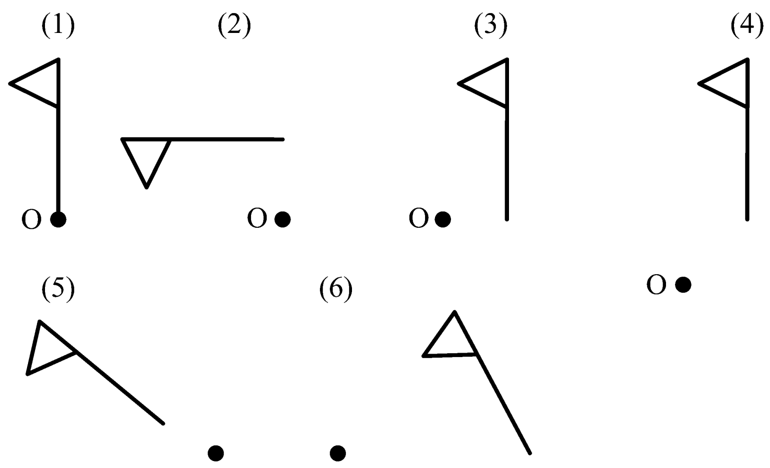
1. A practical hands-on approach is needed. Rotation of shapes is investigated by copying the shape on tracing paper and rotating the paper about the given centre through the given angle. An instruction sheet for pupils is provided on a following page.
2. Clockwise rotations are taken as negative and anticlockwise rotation as positive.
3. Use initially **plain paper** for rotations as this has been found easier before moving to shapes on a grid. The angle of rotation should be restricted to quarter, three quarter and half turns (both clockwise and anticlockwise).
4. Rotations are fully described by (i) centre of rotation (ii) direction of rotation (iii) measure of the angle of rotation.
5. The difficulty of questions on rotation depends on

   (i) ***Position of the centre of rotation.***
   If the centre of rotation is outside the shape to be rotated the question becomes easier than when the centre of rotation is a point on the shape or inside the shape.

   (ii) ***The complexity of the given shape.***
   Points are fairly easy to rotate, shapes of various form are harder. The implication is to emphasize to rotate a shape given on a grid 'point-wise,' i.e., by taking 'one corner' point at a time in case the tracing paper method is not used.

   (iii) ***Position of the object to be rotated.***
   If the original shape is placed such that sides follow grid lines (horizontal and/or vertical) it results in an easier question than when the object is placed sloping with respect to the grid lines.

   The following diagram illustrates questions on rotation in increasing level of difficulty. In all cases the shape is to be rotated through a quarter turn anticlockwise about the centre O.



6. Three types of questions have to be covered
   a) given the centre, the measure of the angle of rotation and the shape, to find the image of the shape when rotated about the centre.
   b) given the centre, the shape and an ‘image’, to find the measure of the angle of rotation.
   c) given a shape and an image of the shape, to find the centre of rotation (if any) and the measure of the angle of rotation.

   The above in combination with (i) the centre of rotation outside, on or inside the shape (ii) a variety of positions of the shape (iii) using grid/coordinates or plain paper.
7. A good amount of practice of rotating of shapes is need.
8. The main properties of a rotation are used for finding the centre of rotation and the angle of rotation other than by inspection (which is the common method for pupils at lower secondary school).

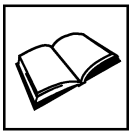
   In a rotation about O in which A maps onto A'
   (i) the angle of rotation is the angle AOA'.
   (ii) the perpendicular bisector of AA' passes through the centre O.

Unit 3, Practice activity 3

1. Design a diagnostic test to find out pupils’ common errors when rotating shapes. Ensure all possible cases are covered: rotation with or without grid, centre of rotation outside, on or inside the shape, variety of positions of the object with respect to the centre of rotation. For each item state the objective: What error is the item to diagnose?
2. Administer the test and analyse the results.

Present your assignment to your supervisor or study group for discussion.

Section B2: How to draw rotations

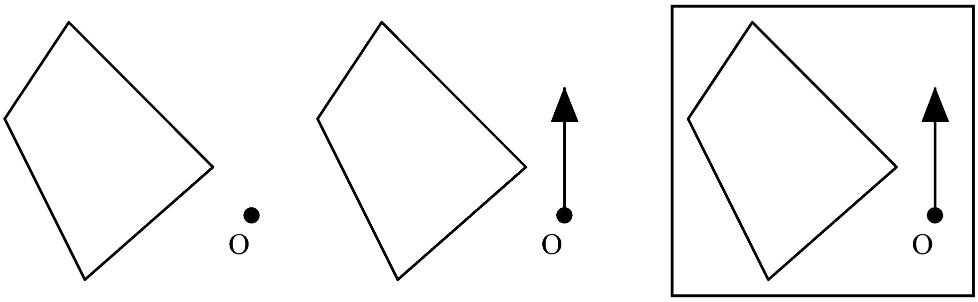


The following is an outline of an instruction sheet for pupils.

Instruction sheet.

If you want to rotate a given shape S about a given centre O follow the following steps.

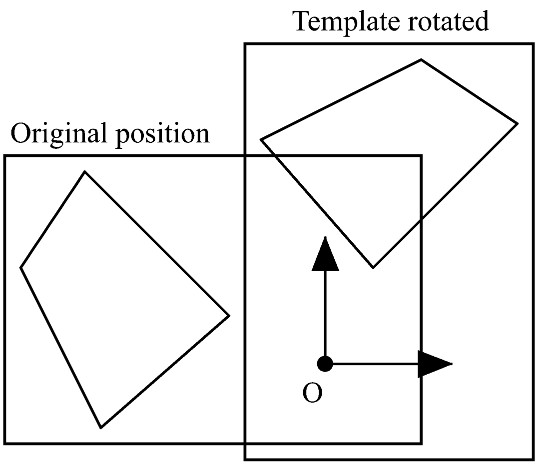
Step 1. At the centre of rotation O fix a 'twelve o'clock' direction.



Step 2: Place tracing paper on top of the shape and copy the shape, the centre O and the 'twelve o'clock' direction. This tracing is your template.

Step 3. Put after tracing the point of your compasses through the template at O (do not move the template as yet).

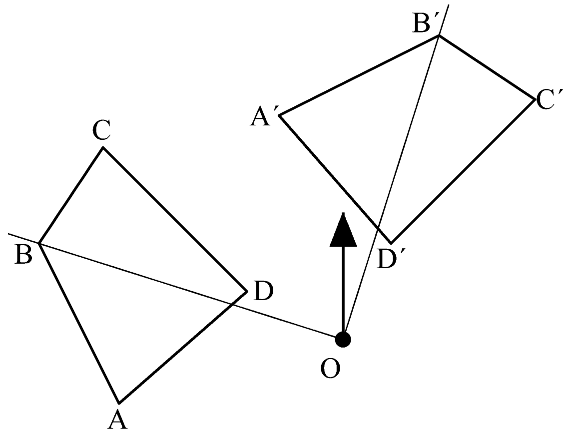
Step 4. Rotate the template through the required angle. You set the 'clock' at the correct time. In the diagram the shape is rotated through $^{-}90^{\circ}$ (90° clockwise, so the two 'hands', on the original paper and on the template are at 'three o'clock').



Step 5. Mark through the template, with another compass or sharp pencil, the position of the vertices of the shape in its new position.

Remove the template and connect the marked points to obtain the shape in its rotated position (the image of the original shape).

In the diagram, the vertices and their images have been labeled. As a check OB and OB′ have been drawn. It is easily checked with a set square that the angle BOB′ is indeed 90º.



Section B3: Examples of type of questions on rotation for the classroom

This section gives examples of the different type of questions to be covered in a classroom situation. Each type of question would need to be extended with similar examples to give sufficient practice and consolidation to the pupils and to cover all possible cases.

The examples cover the following objectives:

Question 1 and 2: Given a shape, the centre of rotation and the angle of rotation pupils are expected to draw the image (using tracing paper).

Question 3: Pupils are to discover the relationship between the coordinates of the original point P(a, b) and its image after rotating about O(0, 0) through (i) 90º (ii) 180º (ii) $^{-}90^{\circ}$ (= 270º).

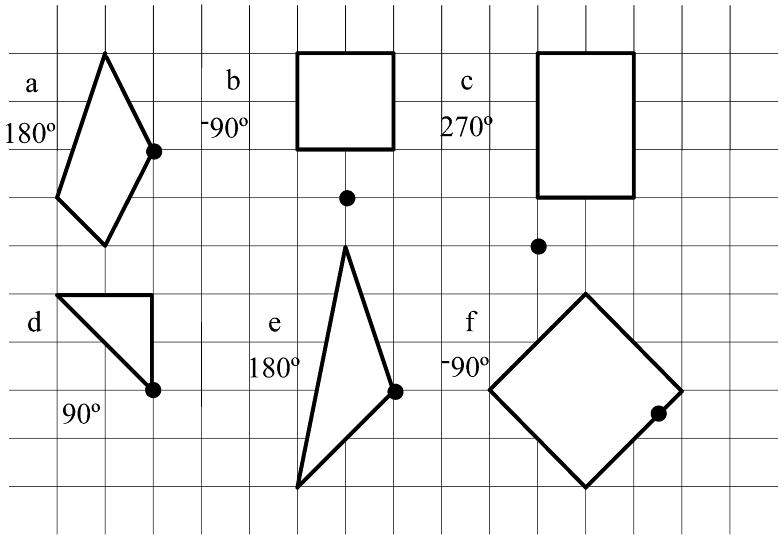
Question 4: Pupils are to find centre and angle of rotation given a shape and its image after a rotation.

Self mark exercise 4

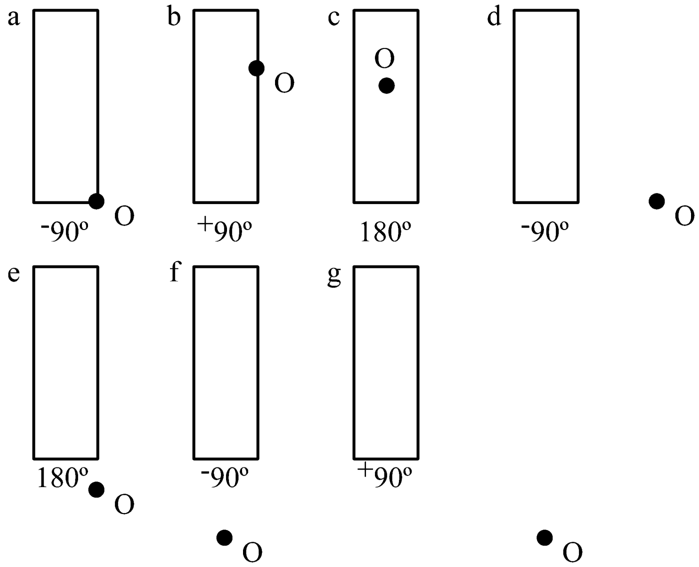
1. Work the questions below.

Check your answers at the end of this unit.

1. Copy the shapes below on squared paper, then use tracing paper to rotate the shape about the centre through the angle indicated.



2. Copy each rectangle and using A4 paper and tracing paper rotate each of the rectangles about O through the angle indicated.



3. a) Plot the points A(3, 2) B(2, $^-3$) C($^-4$, $^-5$) and D($^-6$, 2)

   Rotate each point about O(0, 0) through 90º clockwise. Tabulate the points and their images.

POINT	IMAGE
A(2, 3)	A′(,)

   Can you find a relationship between the coordinates of the point and its image?

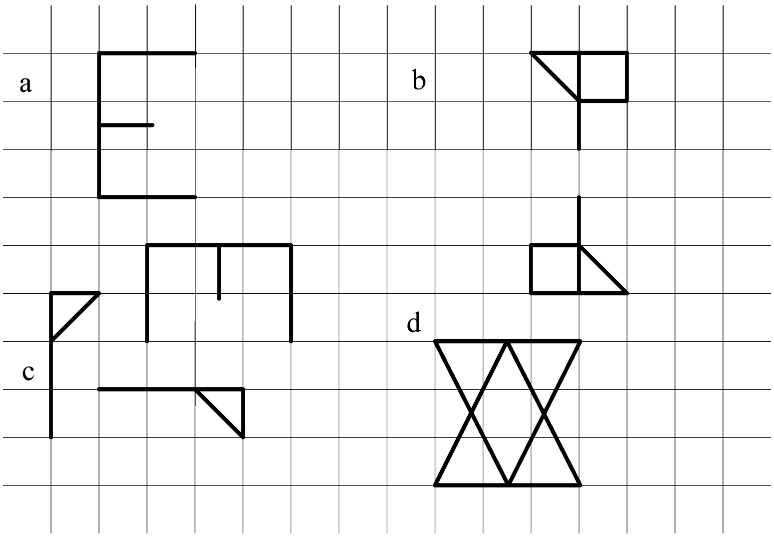
   Check your conjecture for a few more points.

   What would be the image of P(a, b) under a rotation of 90º clockwise ($^-90$º rotation)?

   b) Repeat above for a rotation about O through 180º

   c) Repeat above for a rotation through $^-90$º (which is the same as $^-270$º)

4. Copy the shapes using tracing paper or make cutouts of the shapes. Use your aid to help you to mark the centre of rotation for each pair of congruent shapes. Write down the angle of rotation in each case.



Unit 3, Practice activity 4

1. Choose one of the sample type of questions on rotation you covered above and develop/extend the objective of the question into a complete class activity. Ensure you cover all possible cases with (i) the centre of rotation outside, on or inside the shape (ii) a variety of positions of the shape (iii) using grid/coordinates or plain paper.

   Try it out in your class and write an evaluative report.

2. Write an outline for a coursework assignment on rotation and indicate expected outcomes.

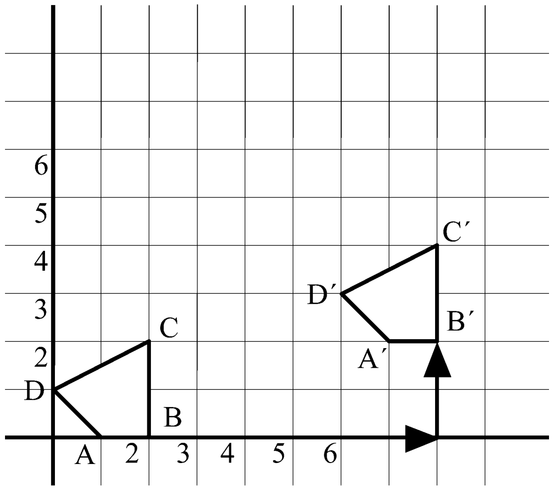
Present your assignment to your supervisor or study group for discussion.

Section C: Translation

This section covers translation of shapes.

Section C1: Points to to keep in mind as a teacher when covering translation

1. Translations or shifts are best described with column vector as illustrated in the diagram



   Shape ABCD (the original) has shifted to A′B′C′D′ (the image).
   A′B′C′D′ is a **translation**—a straight movement without turning—of ABCD.
   The point A moved 6 squares to the right and 2 squares up to A′.
   So did the point B move to B′ , C to C′ and D to D′ (check it!).

2. A short notation is used to describe this translation:

   $$\overrightarrow{AA'} = \overrightarrow{BB'} = \overrightarrow{CC'} = \overrightarrow{DD'} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

   It is called a vector, as it has both a length <u>and</u> a direction.
   Single letters with an arrow above can also be used to indicate vectors.
   For example, $\overrightarrow{A'A}$

3. To move from A′ to A you move six squares to the left and two squares down. It is the opposite of moving from A to A′. The notation used is

   $$\vec{a} = \overrightarrow{A'A} = \begin{pmatrix} ^-6 \\ ^-2 \end{pmatrix}$$

4. The topic does not give as many problems as reflection and rotation. The main point to pay attention to is, as with coordinates, the order in which the numbers are written in the column vector. In $\begin{pmatrix} a \\ b \end{pmatrix}$ the number a represents the horizontal and the number b the vertical shift.

5. Type of questions to cover the topic:

   (i) given a shape and the translation vector, draw the image (or find the coordinates of the vertices of the image).
   (ii) given a shape and its image under a translation, find the translation vector.

Section C2: Examples of type of questions on the topic for the classroom

This section gives examples of the type of questions to be covered with the pupils. The objectives of the questions are that the pupil should be able to

Question 1: Translate a given shape by a given translation vector.

Question 2: Translate a given shape by a given translation vector in order to discover the relationship between the coordinates of the original shape, the translation vector and the coordinates of the image.

Question 3: Describe a translation using a column vector given a shape and its image without coordinates.

Question 4: Describe a translation using a column vector given a shape and its image on a coordinate grid.

Question 5: Describe a translation using a column vector given the coordinates of the original and its image.

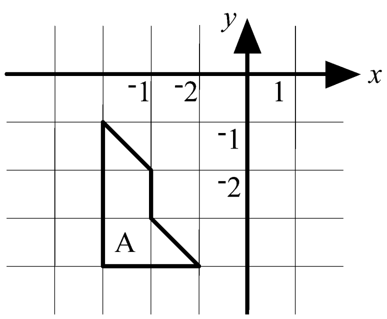
Self mark exercise 5

1. Work the questions below.

Check your answers at the end of this unit.

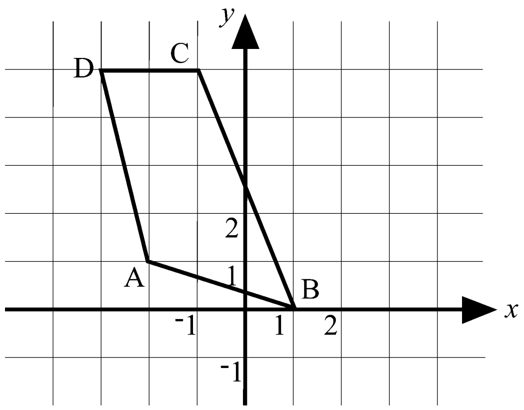
1. Copy on squared paper and translate triangle A by each of the following translation vectors

   a) $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ c) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ d) $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$



2. Copy the quadrilateral ABCD on squared paper and translate it by

a) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ b) $\begin{pmatrix} ^-3 \\ 0 \end{pmatrix}$ c) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ d) $\begin{pmatrix} ^-2 \\ ^-4 \end{pmatrix}$ e) $\begin{pmatrix} 5 \\ ^-6 \end{pmatrix}$

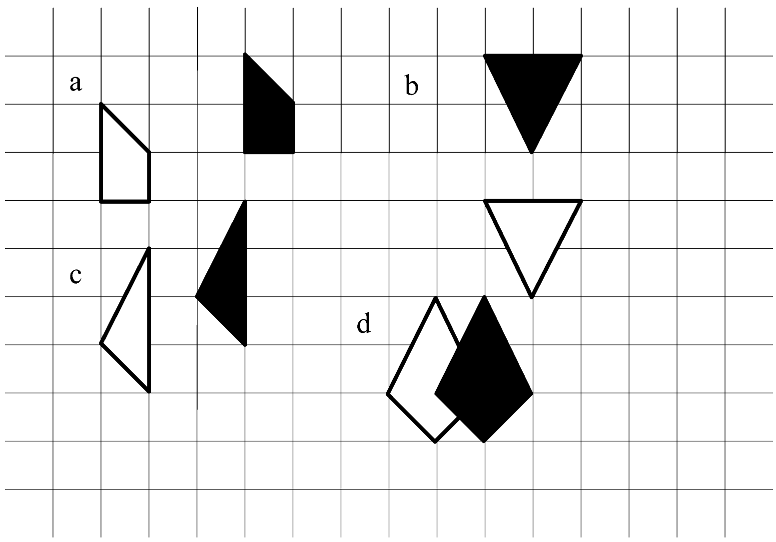


f) Tabulate the coordinates of A, B, C and D and the images.

	a	b	c	d	e
A(-2, 1)					
B					
C					
D					

g) What is the relationship between the coordinates of the original P(x, y), the translation vector v = $\begin{pmatrix} p \\ q \end{pmatrix}$ and the coordinates of the image P′?

3. Use vectors to describe the translation of the white shape (the original) to the position of the black shape (the image).



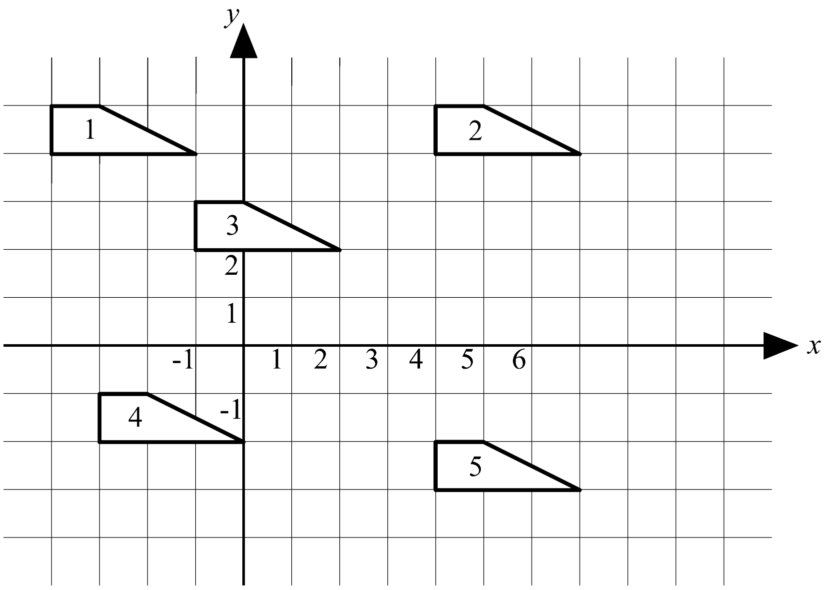
4. Describe fully each of the following translations:

   a) shape 1 onto shape 2 b) shape 1 onto shape 3

   c) shape 1 onto shape 4 d) shape 1 onto shape 5

   e) shape 2 onto shape 3 f) shape 2 onto shape 4

   g) shape 5 onto shape 4 h) shape 5 onto shape 2



5. Find the translation vector given a point and its image.

   a) P(2, 3) ⇒ P′(5, 7)

   b) Q($^{-}5$, $^{-}2$) ⇒ Q′(1, 0)

   c) R(0, 3) ⇒ R′(0, $^{-}5$)

   d) S($^{-}4$, $^{-}6$) ⇒ R′($^{-}2$, $^{-}7$)

   e) T(1, $^{-}8$) ⇒ T′($^{-}4$, $^{-}4$)

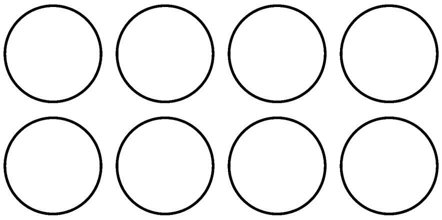
Section C3: Game to consolidate the use of translation vectors

Impala hunt

Needed:

- number cards $^{-}6$, ... $^{-}1$, 1, 2, 3, ... 6
- (2×) game board and counters in different colours for each player
- Cards for "Impala hunt"—for one group copy twice

$^+1$	$^+2$	$^+3$	$^+4$	$^+5$
$^+6$	$^-1$	$^-2$	$^-3$	$^-4$
$^-6$	$^-5$			



Rules:

Shuffle cards and place them face down.

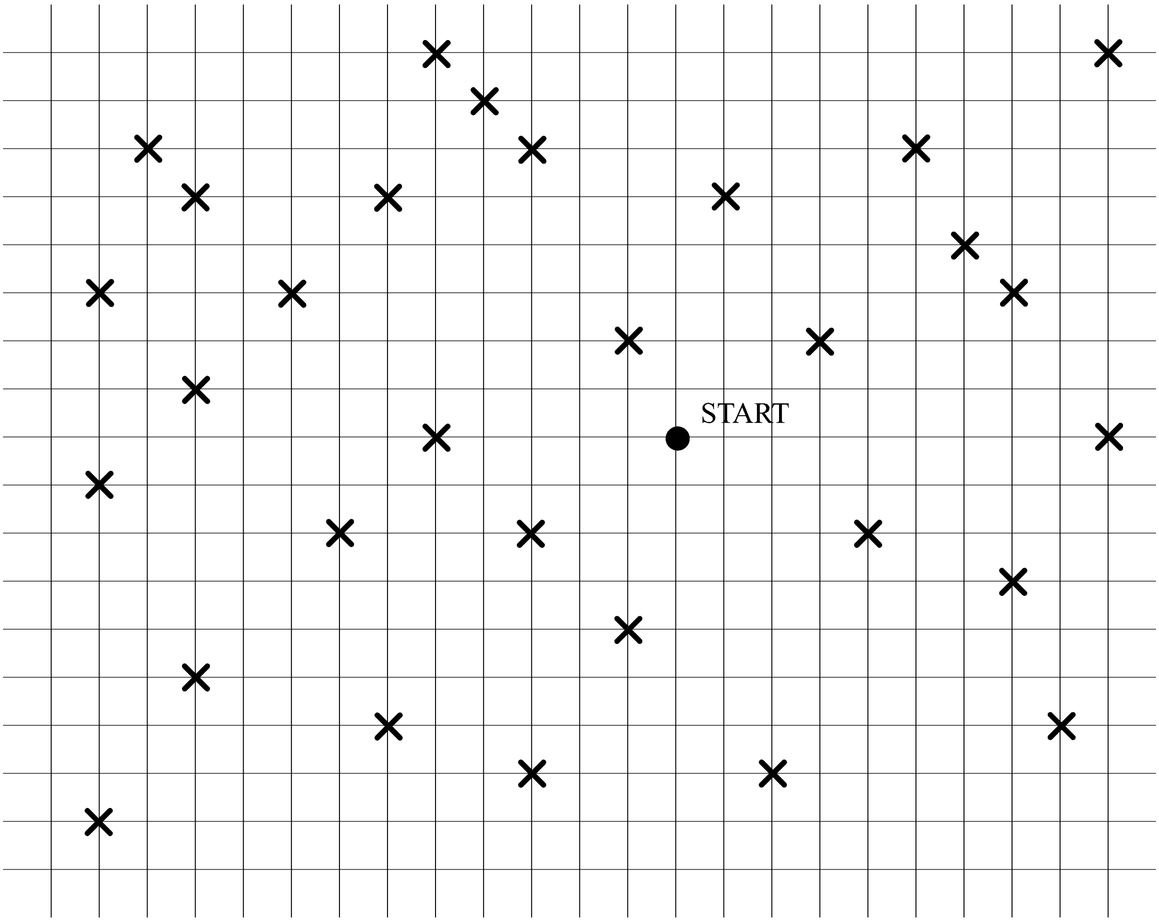
Players take turns in taking two cards and using them to make a vector, e.g., if the two cards are $^-2$ and $^+4$ the vector $\begin{pmatrix} ^-2 \\ ^+4 \end{pmatrix}$ or $\begin{pmatrix} ^+4 \\ ^-2 \end{pmatrix}$ can be made.

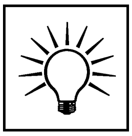
From start move a counter according to the vector made.

Aim: to capture as many impalas as possible (counter remains on the impala).

Winner: player having captured most impalas.

Game Board for "Impala Hunt"





Unit 3, Practice activity 5

1. Make a sufficient number of 'impala' hunt games for your class. Have pupils play 'impala hunt'.

   Try it out in your class and write an evaluative report.

2. Design and describe in detail a game to consolidate reflection, rotation and translation of shapes. Try it out with your class and write an evaluative report.

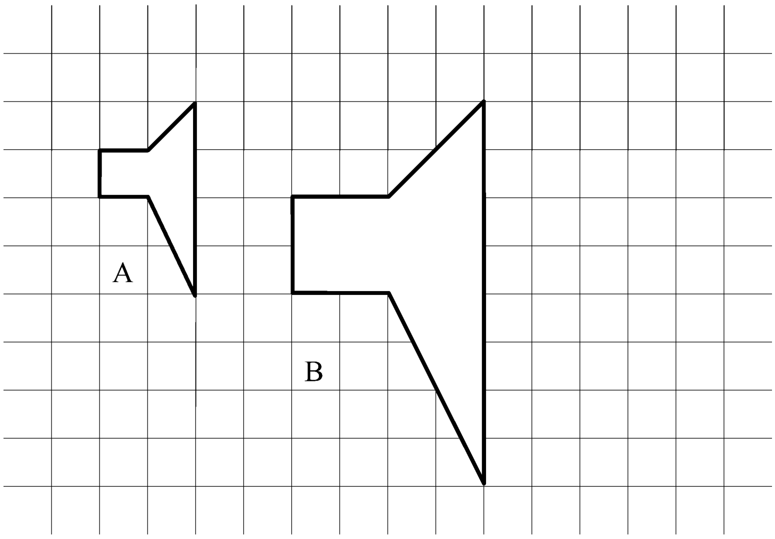
Present your assignment to your supervisor or study group for discussion.

Section D: Enlargement

This section considers a transformation in which the size of the shape changes: enlargements. The word suggests that the image of the shape will be larger in size then the original shape. However this need not be the case. In mathematics the word enlargement is also used if the image is smaller in size then the original.

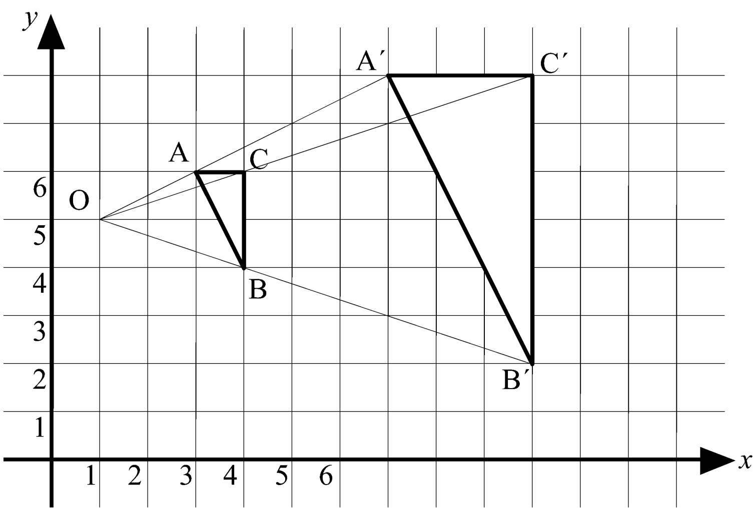
Section D1: Similar and congruent shapes

In translations, reflections and rotation the image and the original shape have the same shape and size. If cut out they fit into each others opening. The shapes are **congruent**.



Shape A and B have the same shape but are different in size. The length measurements in shape B are twice those in shape A. B is a **scale diagram** from A by a **scale factor** 2. The shapes A and B are similar. You could say that for congruent shapes the scale factor is 1.

This diagram illustrates the enlargement of triangle ABC from centre O by scale factor 3. OA′ = 3OA, OB′ = 3OB and OC′ = 3OC



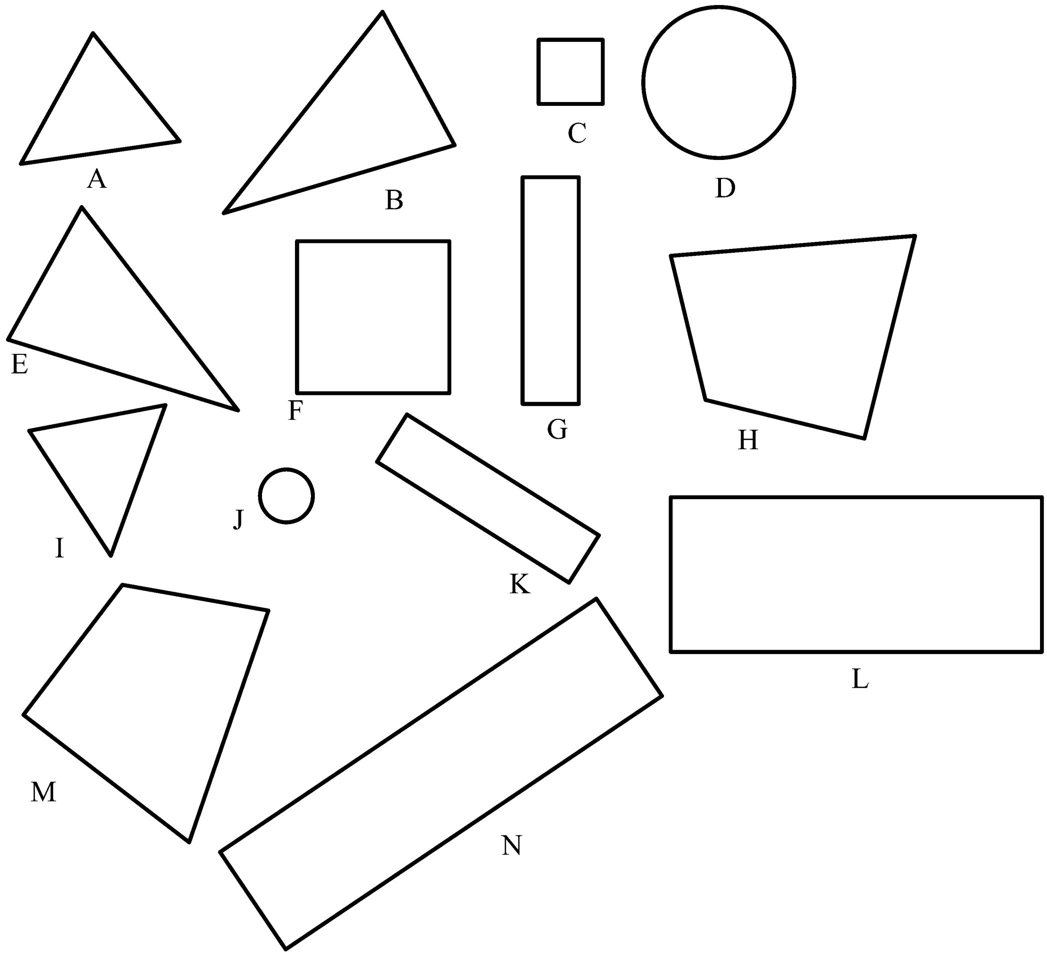
The mathematical meaning of similar is concise: in similar shapes corresponding lengths have the same ratio (the scale factor) and corresponding angles are equal in size. In day to day use “similar” is used in a less strict way: “about the same looking”.

A **sorting** activity is needed to get the concept of similar clear in the minds of the pupils. On this activity pupils should work in groups to discuss whether or not shapes are similar. Real objects and photographs of objects should be used apart from ‘abstract’ diagrams like the ones below for a discussion exercise for the pupils.

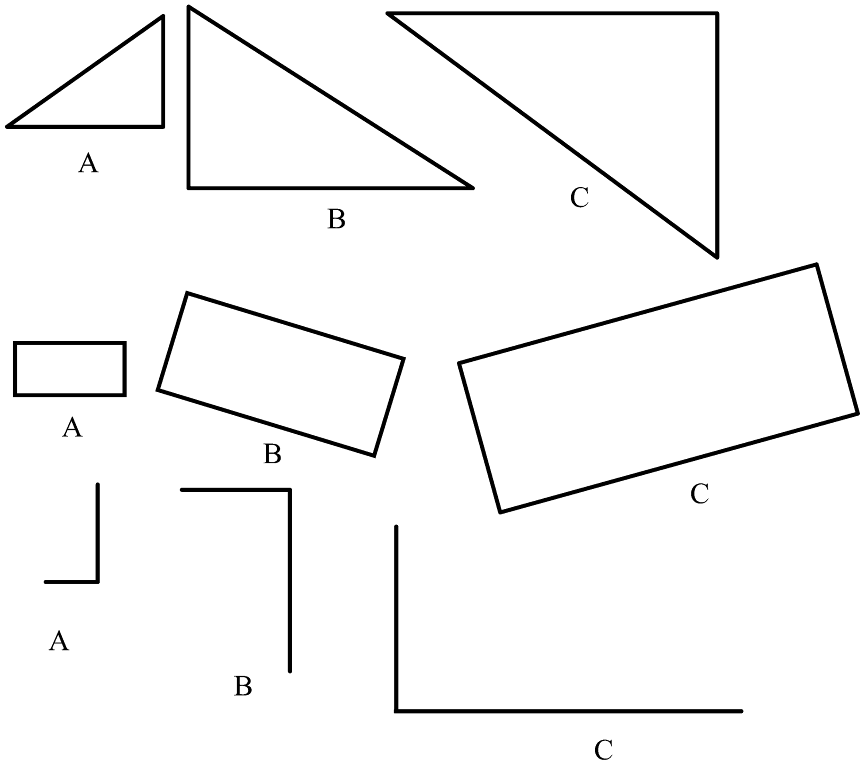
Discussion exercise 1

Work through this exercise as a group. First each member of the group **thinks**. Next, **share** your thoughts. Discuss until you agree as group. Listen carefully to each others’ arguments.

1. Identify in the diagram the shapes that are congruent and the shapes that are similar.



2. Which of the shapes B or C is NOT similar to A? Justify your answer.



3. Discuss whether the following statements are true or false. Justify your answers using diagrams to give examples or non-examples to support your argument.

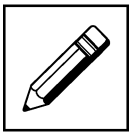
   a) Rectangles are always similar to each other.

   b) All squares are similar.

   c) All equilateral triangles are congruent, as their angles have equal sizes of 60º.

   d) Regular polygons are similar.

   e) Regular *n* sided polygons are similar.



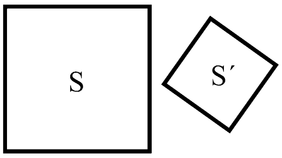
Self mark exercise 6

1. Answer the questions in discussion exercise 1 above.

Check your answers at the end of this unit.

Section D2: Learning about enlargments

As is a transformation, an enlargement is determined by (i) the centre of enlargement (ii) the scale factor of the enlargement. The concept of enlargement is, as most concepts in mathematics, developed gradually. A scale diagram S′ of a shape S need not to be a (direct) enlargement in the mathematical sense as there might not be a centre.



The two squares S and S′ are similar, but S′ cannot be obtained by enlarging S as there is no centre of enlargement. (S′ can be obtained from S by more than one linear transformation applied in succession e.g. enlargement followed by rotation followed by translation.) The confusion is that a scale drawing of an object is often called an enlargement although there might not be a centre. As for linear transformations, enlargements require a centre of enlargement. It is advisable to use the word enlargement only if S′ can be obtained from S by enlarging from a centre C. If there is no centre, S′ is not an enlargement but a shape similar to S with a certain scale factor.

Two polygons are similar when they are equiangular and corresponding sides are in the same ratio. Equiangular polygons are not necessarily similar (e.g. a 3 cm by 3 cm square and a 6 cm by 8 cm rectangle are equiangular but corresponding sides do not have the same ratio and hence the two quadrilaterals are not similar). In the case of a triangle, however, if two triangles are equiangular then they are similar (and corresponding sides are in the same ratio). Also: If two triangles have corresponding sides in the same ratio then the two triangles are similar (and are equiangular).

Write down the different stages in the development of the concept 'enlargement' often covered in different year groups. What aspect of enlargement do you cover in which year?

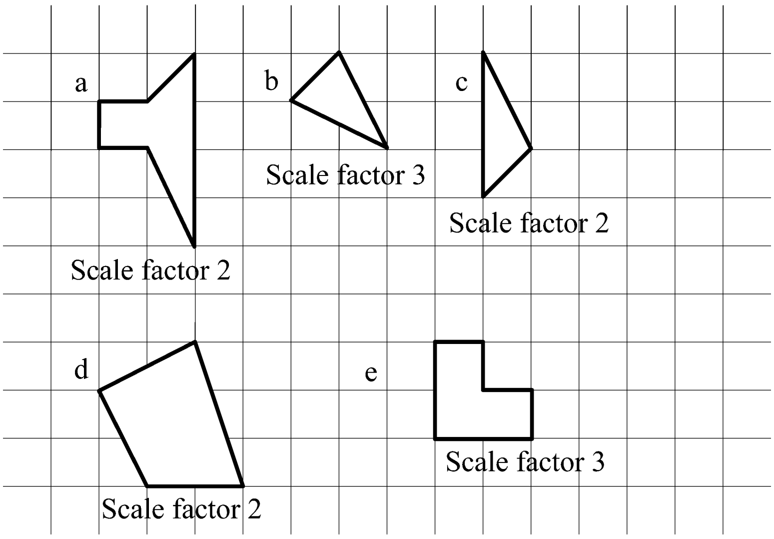
Compare your list with the following sequence. Your order might differ at some points but the different aspects should be presented to pupils in a variety of activities.

(i) identifying similar and congruent shapes (see discussion exercise 1).

(ii) making a shape similar to a given shape by a whole number scale factor greater than 1 without reference to a centre. (The similar shape drawn might or might not be an enlargement depending on whether or not a centre can be found.)

Example

Copy each of the following shapes on squared paper and enlarge by the scale factor indicated.



(iii) finding a scale factor by measuring corresponding lengths, the scale factor being a whole number greater than 1

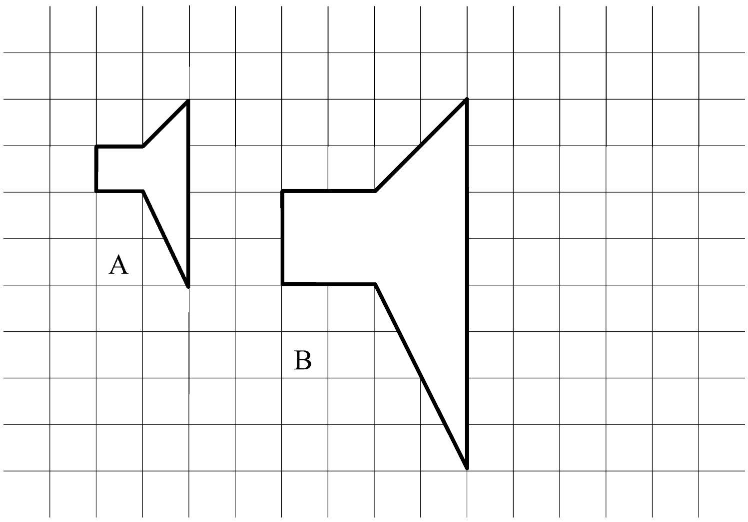
To find the scale factor you find the ratio of the length of any two corresponding line segments in image and original

$$k = \frac{\text{length of image segment}}{\text{length of original segment}}$$

If the object and image are at opposite sides of the centre of enlargement you must place a negative sign with the above ratio.

Example

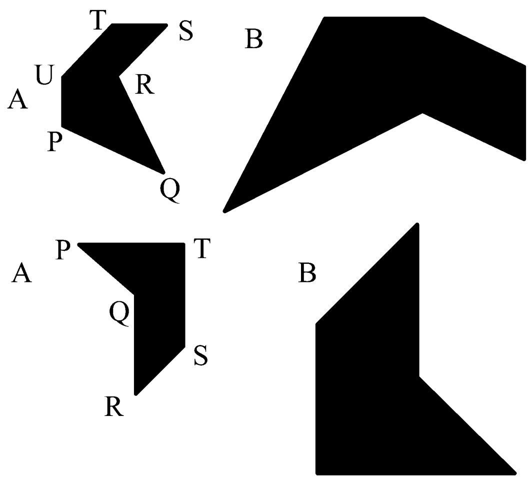
Shape B is similar to shape A. By taking appropriate measurements, find the scale factor.



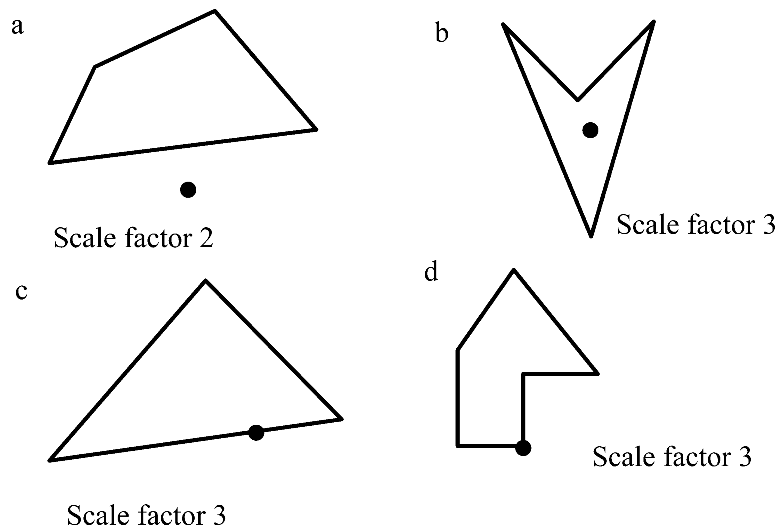
(iv) enlarging from a centre O with whole number scale factors greater than 1 with and without the presence of a grid, with the centre outside, on a side, at a vertex and inside the shape

Examples

Copy the following shapes on plain paper and enlarge each from the marked "A" or "B" centre by the scale factor that you indicate.



Copy each of the following shapes on square cm paper and enlarge the shape from O with the scale factor indicated. Label the vertices of the object A, B, C, … and its image A′, B′, C′, ...



(v) discovering the relationships

a) In an enlargement, the ratio (lengths of image of line segment): (lengths of line segment) is constant and equal to the scale factor.

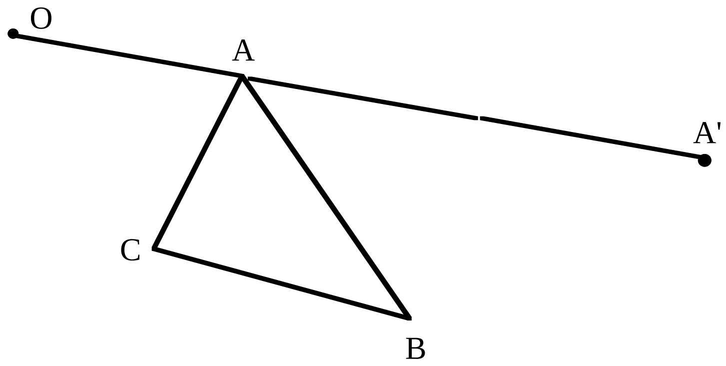
b) In an enlargement, a line segment and its image are parallel.

c) In an enlargement, the line joining a point with its image passes through the centre of enlargement.

Example

a) On paper, draw a diagram similar to the one below. Join the point O (the centre of enlargement) with A, B and C.

b) Measure OA. Using the scale factor 3, mark A′ such that OA′ is $3 \times$ OA.



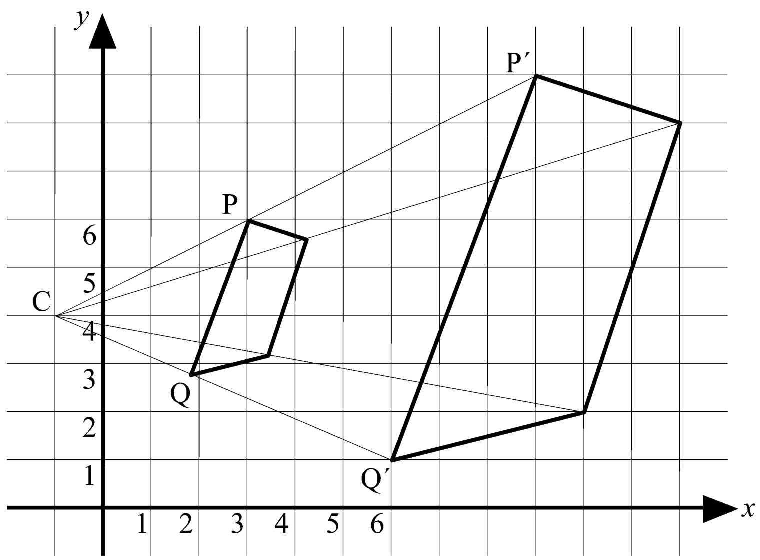
c) Repeat for the point B to find a point B′ and for C to find a point C′. Join A′B′C′ to form triangle A′B′C′ an enlargement of triangle ABC by scale factor 3. Check that the lengths of the sides of triangle A′B′C′ are three times the length of the corresponding sides of triangle ABC.

d) Compare the directions of AB and A′B′, of BC and B′C′ and of AC and A′C′. What do you notice?

e) Start with another shape, for example a quadrilateral PQRS, and enlarge it from a centre O by a scale factor 2. Check your observation of (d).

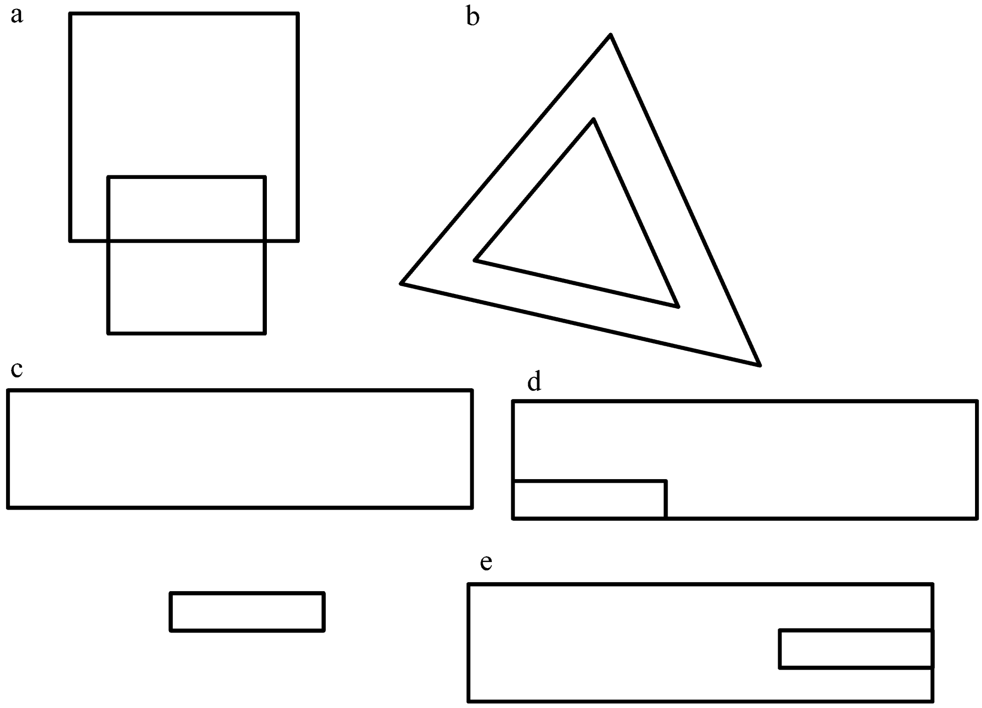
(vi) application of the property (c) listed in (v) above: finding the centre of enlargement. The shapes might be given on a grid or not. The centre can be positioned outside, on a side, at a vertex or inside the shape.

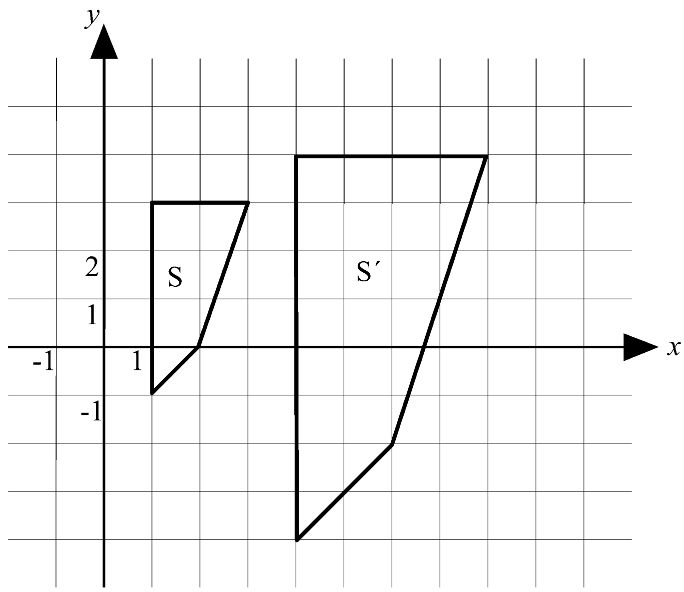
To find the centre of enlargement connect points with their image. For example P with P′ and Q with Q′. The point of intersection of PP′ and QQ′ is the centre C($^-1$, 4) of enlargement as illustrated in the diagram.



Examples

Find the centre of enlargement in each of the following figures.

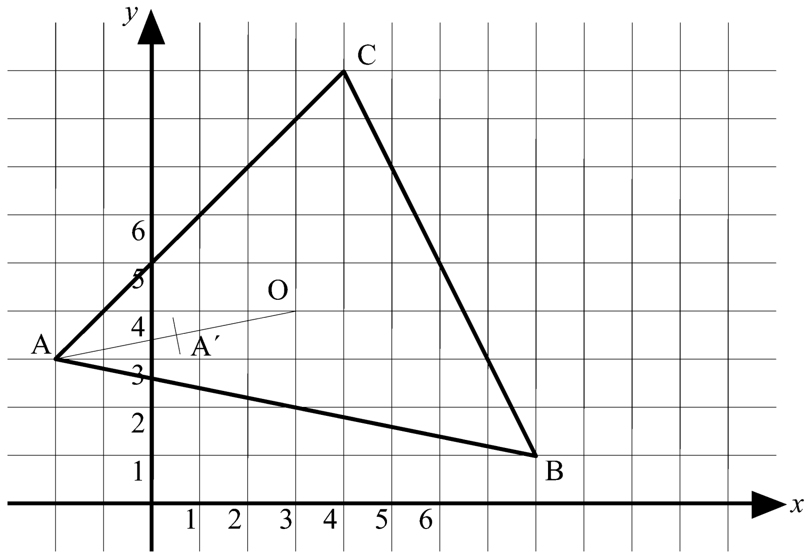




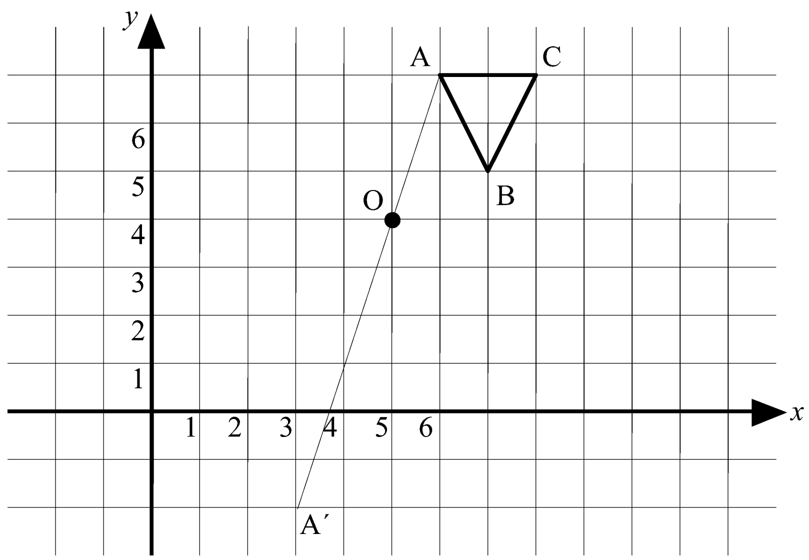
(vii) enlarging by fractional ($^{-}1 < k < 1, k \neq 0$) and negative scale factors from a centre O. Enlargement by fractional scale factor ($^{-}1 < k < 1$, $k \neq 0$) gives an image that is smaller in size, yet it is called enlargement. With negative scale factors the image and objects are at opposite sides of the centre of enlargement.

Example

Enlarge triangle ABC from centre O by scale factor $\frac{1}{2}$. Connect O with A and find A′ on OA such that OA′ = $\frac{1}{2}$ OA. Repeat for the other points.



Enlarge triangle ABC from centre O with scale factor ⁻2 . Connect O with A, produce AO to A′ such that OA′ = 2 OA. Repeat for the other points.



(viii) enlarging on a grid from O(0, 0) and discovering that the coordinates of the original P(x, y) map onto P′(kx, ky) under an enlargement centre O, scale factor k.

Example

a) On a square grid draw a coordinate system and plot the points A(2, 1), B(1, 3) C(⁻1, ⁻1) and D(1, ⁻2). Complete the quadrilateral ABCD.

Enlarge ABCD with centre O(0, 0) by scale factor 2, 3 and 4 respectively.

Write down the coordinates of the image points in the table below.

Original	**A(2, 1)**	**B(1, 3)**	**C(⁻1, ⁻1)**	**D(1, ⁻2)**
Scale factor 2				
Scale factor 3				
Scale factor 1.5				
Scale factor 10				
Scale factor 50				
Scale factor *n*				

b) Look at the pattern. Can you easily find the coordinates of the image points if you know the scale factor?

Complete the last three rows in the table.

Self mark exercise 7

1. Work the example questions in the preceding outline.
2. Look at the following statements and decide whether they are always true, sometimes true or always false. Justify your answer illustrating with examples.

   a) If S′ is an enlargement of S then S′ and S are similar.

   b) If S′ and S are similar then S′ is an enlargement of S.

   c) All squares are enlargements of each other.

   d) All squares are similar.
3. Enlarge a shape S by a scale factor p to give S′. Enlarge S′ using a scale factor q to give you S″. Investigate the relationship between S and S″.

Check your answers at the end of this unit.

Transformation geometry calls for a practical method: pupils moving, flipping, rotating shapes. Enlargement concepts can be developed using an elastograph to enlarge shapes.

The elastograph is a simple device for enlarging (which includes reducing in size: scale factor s of enlargement is in the range $1 < s < 1$, $s \neq 0$) diagrams. It can be used as a practical method for the introduction of scale factors (i.e. ratios of enlargement).

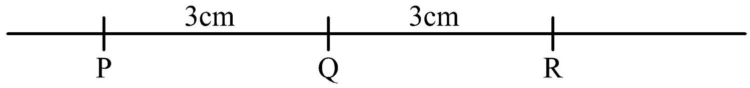
The length of elastic used should be capable of stretching to at least three times its natural length in order to produce reasonable sized drawings.

Below you find an outline for a worksheet for the pupils for use in the classroom as a first introduction to enlargement.

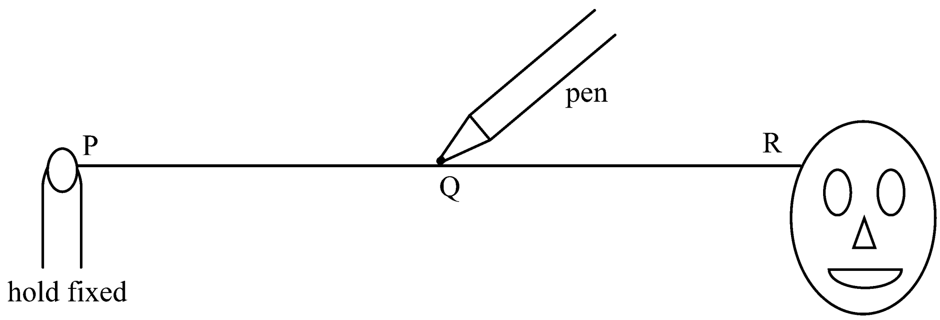
Worksheet.

Work in pairs.

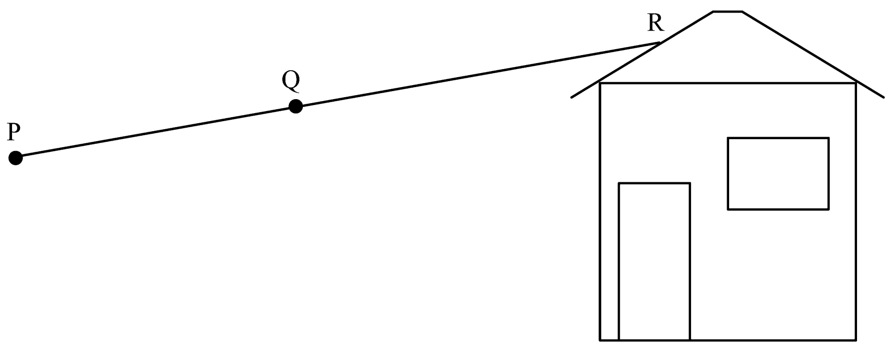
1. To make an elastograph you take a piece of elastic and make 3 marks on it 3 cm apart with a ball point pen. Call the points P, Q and R.

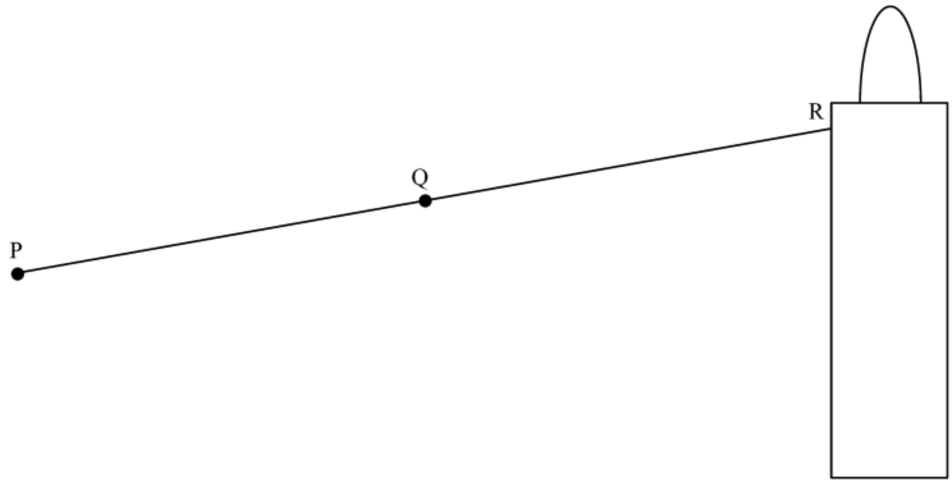


2. One person holds P fixed and moves R around the diagram of the face. The other person marks the path of Q.



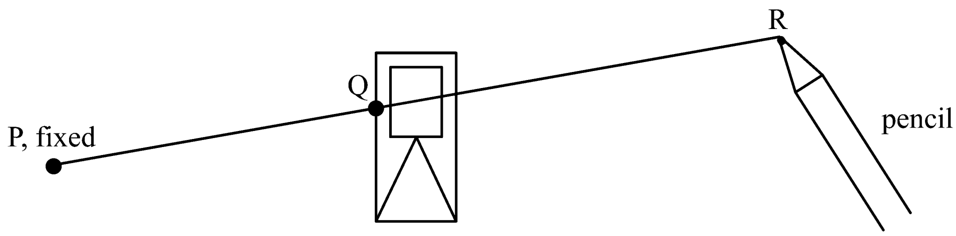
3. Do the same for the following shapes:





4. What do you notice about the diagram you have drawn as compared to the given diagram?
5. Take another piece of elastic and mark P and next Q, 3 cm from P and R, 6 cm from Q.

   Use your elastograph to enlarge the shape below. Keep P fixed, place the pencil at R and stretch such that Q follows the shape.



6. How many times larger is your diagram than the original?
7. If you place the first mark Q, 3 cm from P where are you to place R if you want to enlarge the original 4 times? 5 times?
8. Make your own design and enlarge it.

Unit 3, Practice activity 6

1. Choose one of the stated aspects (i - viii) to be covered under enlargement and develop/extend the objective of the example question into a complete class activity. Ensure you cover all possible cases with (i) the centre of enlargement outside, on or inside the shape (ii) a variety of positions of the shape (iii) using grid/coordinates or plain paper (iv) using a variety of scale factors.

   Try it out in your class and write an evaluative report.
2. Try out in a class the practical activity with the elastograph and compare with the more abstract method used in 1. Write an evaluative report specifically focussing on the reaction of the pupil. Which of the two methods gave a higher pupil involvement and led to greater enjoyment? Why was that the case?

Present your assignment to your supervisor or study group for discussion.

Summary

Unit 3 has continued the practical approach to help your students understand geometrical transformations. By taking them from relatively concrete concepts like “enlargement” to the much more abstract “congruent” and “similar,” using practical manipulations throughout, it should prepare them for more complex geometry in subsequent years.

Unit 3: Answers to self mark exercises

Self mark exercise 1

Several methods can be used to obtain the images.

(i) use of tracing paper: draw the original and the mirror line on the tracing paper and next flip over the tracing paper and place it with the mirror lines coinciding. Position of image can now be marked with sharp pencil or point of compass.

(ii) use of ruler/set square/compasses to draw from each point lines perpendicular to the mirror line and measure (using the compasses) distances such that the original point is the same distance from the mirror line as its image.

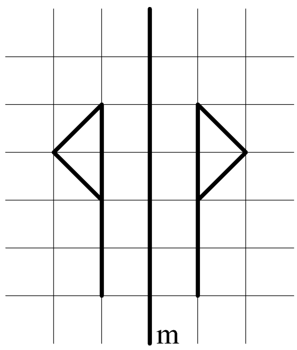


Figure 1

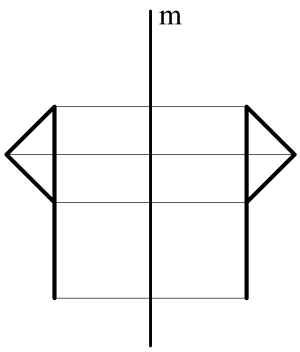


Figure 2

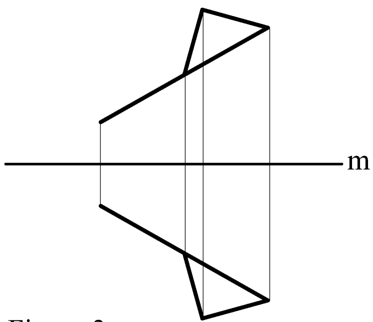


Figure 3

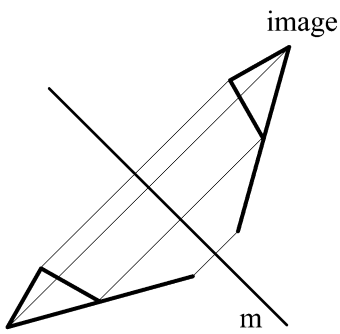


Figure 4

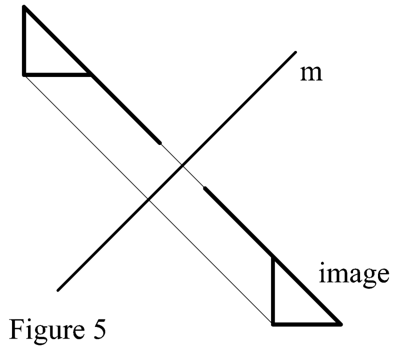


Figure 5

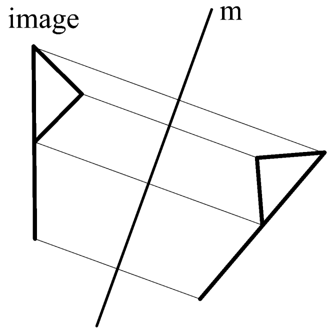


Figure 6

Self mark exercise 2

Worksheet 1

3. 90º PP′ is perpendicular to the mirror line.
4. PS = PS′. Distance from P to the mirror line is equal to the distance from P′ to the mirror line.
5. Points on the mirror line are their own image.

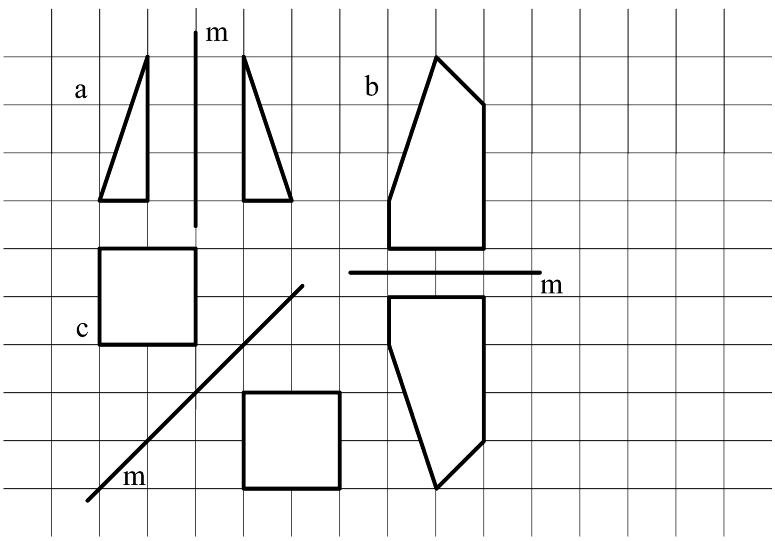
Worksheet 2

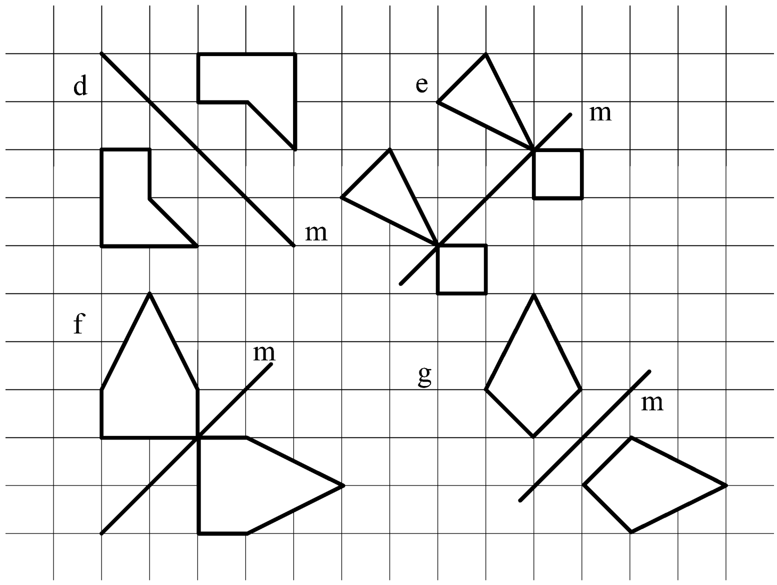
1. A line and its image meet on the mirror line.
2. AB and A′B′ do not meet but are parallel.

   If a line is parallel to the mirror line the image of the line is also parallel to the mirror line.

Worksheet 3

1.





2. Corrections are needed for a, c , d, e, f and h.

Worksheet 4

1. $(^-3, 2)$ $(5, 2)$ $(2, 3)$ $(^-7, ^-2)$ $(4, ^-5)$ $(1, 3)$ $(^-4, 6)$
   $(^-5, ^-6)$ $(3, b)$ $(^-a, 4)$ $(^-a, b)$ $(^-2a, 3b)$
2. $(3, ^-2)$ $(^-5, ^-2)$ $(^-2, ^-3)$ $(7, 2)$ $(^-4, 5)$ $(^-1, ^-3)$ $(4, ^-6)$
   $(5, 6)$ $(a, 3)$ $(4, ^-b)$ $(a, ^-b)$ $(2a, ^-3b)$
3. $(2, 3)$ $(2, ^-5)$ $(3, ^-2)$ $(^-2, 7)$ $(^-5, ^-4)$ $(3, ^-1)$ $(6, 4)$
   $(^-6, 5)$ $(b, ^-3)$ $(4, a)$ (b, a) $(3b, 2a)$
4. $(^-2, ^-3)$ $(^-2, 5)$ $(^-3, 2)$ $(2, ^-7)$ $(5, 4)$ $(^-3, 1)$ $(^-6, ^-4)$
   $(6, ^-5)$ $(^-b, 3)$ $(^-4, ^-a)$ $(^-b, ^- a)$ $(^-3b, ^-2a)$
5. $(^-a, b)$ $(c, ^-d)$ (f, e) $(^-h, ^-g)$
6. $(^-3, ^-2)$ $(^-5, 20)$ $(3.7, ^-2.3)$ $(2, ^-9)$ $(5.6, 3.5)$ $(32, 45)$
   $(^-8, 12)$ $(^-7.3, 4.7)$

   $x = y$ $x = 0$ $y = 0$ $x = ^-y$

5. If the two parallel (vertical) lines are distance d apart (if the lines have as equation $x = a$ and $x = b$ and the first reflection is in $x = a$ followed by the reflection in $x = b$ then $d = b - a$) the double reflection is equivalent to a translation by the vector $\begin{pmatrix} 2d \\ 0 \end{pmatrix}$

6. a) $(6 ^-a , b)$ b) $(a, 6 - b)$ c) $(3 - b, 3 - a)$ d) $(3 + b, a - 3)$

7. $(\frac{y - n}{m}, mx + n)$ provided $m \neq 0$

Self mark exercise 3

1. a) $x = 1\frac{1}{2}$ b) $x + y = 8$ c) $y = 2\frac{1}{2}$

   d) $x = 2\frac{1}{2}$ e) $y = x$ f) $y = x$

2. a) $x = 2$ b) $y = 1$ c) $x + y = 2$ d) $y - x = 3$

Self mark exercise 4

3. a) Image of (a, b) is $(b, {}^{-}a)$

   b) Image of (a, b) is $({}^{-}a, {}^{-}b)$

   c) Image of (a, b) is $({}^{-}b, a)$

4. a) $^{-}90^{\circ}$ or 90° depending which of the two shapes is taken as the original

   b) 180°

   c) $^{-}90^{\circ}$ or 90° depending which of the two shapes is taken as the original

   d) 180°

Self mark exercise 5

2. f)

	a	b	c	d	e
A($^{-}2$, 1)	($^{-}2$, 4)	($^{-}5$, 1)	(0, 6)	($^{-}4$, $^{-}3$)	(3, $^{-}5$)
B(1, 1)	(1, 4)	($^{-}2$, 1)	(3, 6)	($^{-}1$, $^{-}3$)	(6, $^{-}5$)
C($^{-}1$, 5)	($^{-}1$, 8)	($^{-}4$, 5)	(1, 10)	($^{-}3$, 1)	(4, $^{-}1$)
D($^{-}3$, 5)	($^{-}3$, 8)	($^{-}6$, 5)	($^{-}1$, 10)	($^{-}5$, 1)	($^{-}2$, $^{-}1$)

   g) $P'(x + p, y + q)$

3. a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ b) $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ d) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

4. a) $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ {}^{-}2 \end{pmatrix}$ c) $\begin{pmatrix} 1 \\ {}^{-}6 \end{pmatrix}$ d) $\begin{pmatrix} 8 \\ {}^{-}7 \end{pmatrix}$

   e) $\begin{pmatrix} {}^{-}5 \\ {}^{-}2 \end{pmatrix}$ f) $\begin{pmatrix} {}^{-}7 \\ {}^{-}6 \end{pmatrix}$ g) $\begin{pmatrix} {}^{-}7 \\ 1 \end{pmatrix}$ h) $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$

5. a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ b) $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ c) $\begin{pmatrix} 0 \\ ^-8 \end{pmatrix}$ d) $\begin{pmatrix} 2 \\ ^-1 \end{pmatrix}$ e) $\begin{pmatrix} ^-5 \\ 4 \end{pmatrix}$

Self mark exercise 6

1. Congruent B and E, A and I, G and K, H and M

   Similar D and J, J and F, K and N, G and N <u>and</u> the above congruent shapes (congruent shapes are similar!)

2. C C B

3. b, e true, a, c, d false

Self mark exercise 7

1. No problems expected in the sample questions

2. a) true b) false c) false d) true

3. S″ is an enlargement of S by scale factor pq. This is rather straight-forward if both enlargements have the same centre (say O).

   $(x, y) \Rightarrow (px, py) \Rightarrow (pqx, pqy)$

   If the first enlargement has centre A(a, b) and the second another centre B(c, d), the resultant will be an enlargement with factor pq from a centre

   C($c + apq + cq - aq, d + bpq + dq - bq$).

Unit 4: Transformations II

In this unit you will have a further look into transformation. The transformations considered are all linear transformations of the plane onto itself. These are the transformations that can be represented by linear expressions. The point P(x, y) maps onto P′($ax + by + c$, $dx + ey + f$), where a, b, c, d, e, and f are constants. When $c = f = 0$ the origin O(0,0) maps onto itself—it is an invariant point. The transformation can be written in this case as $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix}$ where T is the matrix $\begin{pmatrix} a & b \\ d & e \end{pmatrix}$. If c and or f is not 0 the transformation does not leave O fixed. A translation is an example of such a transformation.

Purpose of Unit 4

The purpose of this unit is to extend your knowledge on transformations by considering combination of transformations. If S and T represent transformations and A is a shape, you can look at the image of A under S, written as S(A), and apply to the image the transformation T. The notation used is TS(A). Take note of the order of writing!

This unit also looks at how linear transformations can be described using matrix notation. A point in two dimensional space can be represented by a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Transformation onto a point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ can be described by a matrix multiplication $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathrm{T}\begin{pmatrix} x \\ y \end{pmatrix}$ where T is the transformation matrix and the origin O is a fixed point i.e. maps onto itself.

The teaching emphasis in this unit is on investigative work and looks at how to assess investigative work. You will be asked to investigate the above topics.

Objectives

When you have completed this unit you should be able to:

- apply a combination of two transformation to a shape
- describe the single transformation equivalent to two successive transformations
- use the notation PQ(S) meaning the transformation Q followed by P applied to S
- investigate commutativity of two successive transformations
- use matrices to describe transformation
- investigate the effect of basic transformation and combination of two of them
- use the given assessment scheme to assess pupils′ investigative work

Time

To study this unit will take you about 10 hours. Trying out and evaluating the activities with your pupils in the class will be spread over the weeks you have planned to cover the topic.

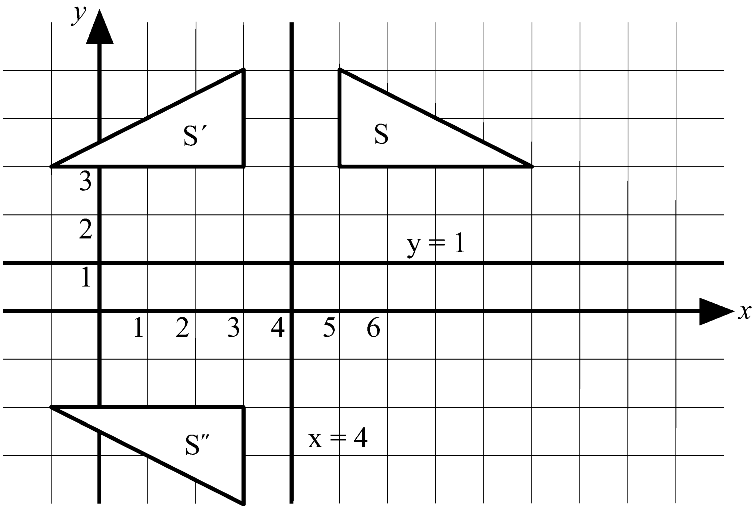
Section A1: Combined transformations

Transformation can be combined: one transformation followed by another transformation. The resulting transformation can frequently be described by an equivalent single transformation.

Example 1

The shape S is reflected in the line $x = 4$ to give the image S′. S′ is reflected in the line with equation $y = 1$ to give as image S″.

Describe the transformation that maps S onto S″.



In the diagram the reflections have been drawn. S maps onto S″ by a rotation through 180° about the centre (4, 1).

It is convenient to denote transformation by using capital letters. **A** could denote "reflection in the line with equation $x = 4$" and **B** could denote "reflection in the line with equation $y = 1$".

Performing **A** on S to give S′ is written as **A**(S) = S′.

B(S′) means "perform the transformation **B** on shape S′." **B**(S′) = S″

The combined transformation is written as **BA**(S) = S″.

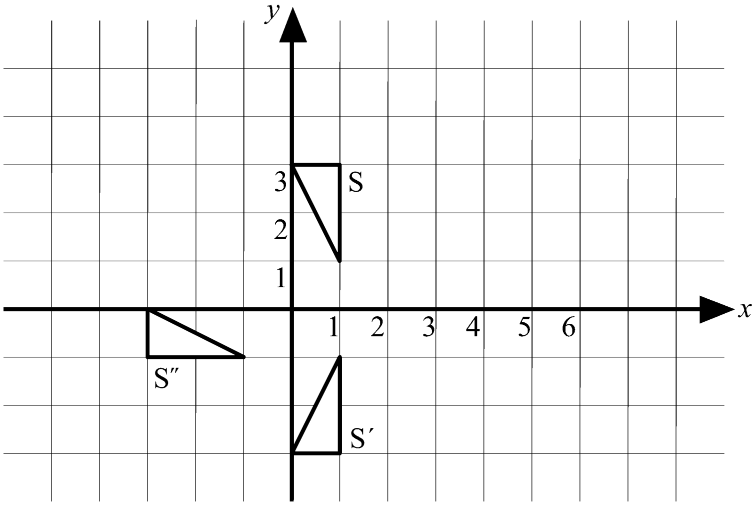
It is important to notice the order **BA**(S) means do first **A** and then **B**.

Example 2

The triangle S is reflected in the x-axis ($y = 0$) to give the image S′. S′ is rotatated about O through $^{-}90^{\circ}$. The image is S″.

Describe the single transformation that maps S onto S″.

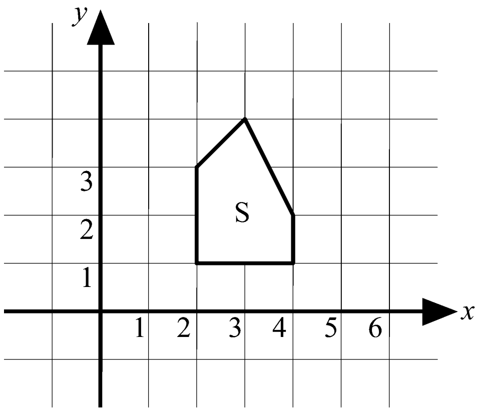
The diagram illustrates the transformations described.



If **A**(S) = S′ and **B**(S′) = S″ then **BA**(S) = S″ and **BA** is equivalent to a reflection in the line with equation $x = {}^{-}y$.

Self mark exercise 1

1. **A** is the transformation reflect in the line with equation $x = 4$
   B is the transformation reflect in the line with equation $x = 2$



   a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

   b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

   c) Compare the positions of the final shape in a and b. Does the order matter?

2. **A** is the transformation reflect in the x-axis.
   B is the transformation rotate about O through 180º.

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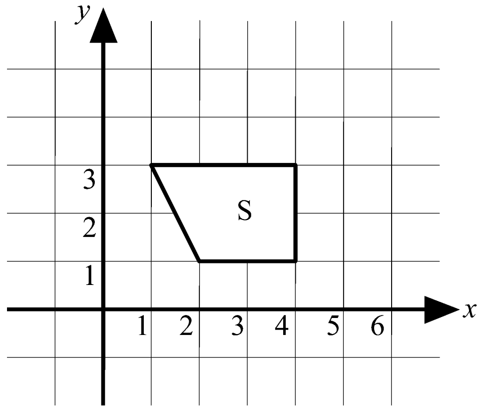
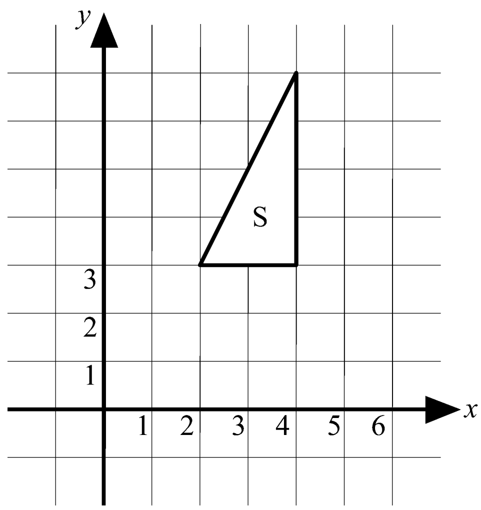


a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

c) Compare the positions of the final shape in a and b. Does the order matter?

3. **A** is the transformation reflect in the y-axis.
   B is the transformation reflect in the line with equation $x = {}^{-}y$.



a) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **BA**(S).

b) Copy S on squared grid paper and find the single transformation equivalent to the combined transformation **AB**(S).

c) Compare the positions of the final shape in a and b. Does the order matter?

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4. The vertices of triangle PQR have coordinates P(5, 0) Q(5, __) R(2, $^-2$).
   a) Plot the points and draw the triangle on grid paper. Call the triangle T.
   b) The triangle T is rotated about O, through 90º; its image is U.
   c) Reflect triangle U in the line with equation $x = {}^-1$. The image obtained is called V.
   d) Reflect triangle T in the line with equation $y = 1$. The image is W.
   e) Describe a single transformation that would map W onto V.
5. The vertices of triangle PQR have coordinates P($^-1$, 2) Q($^-$______ and R($^-3$, 5).
   a) Plot the points and draw the triangle on grid paper. Call the triangle A.
   b) Reflect triangle A in the line with equation $y = {}^-1$. The image is B.
   c) Reflect triangle A in the line with equation $x = {}^-y$. The image obtained is called C.
   d) Reflect triangle B in the line with equation $x = {}^-y$. The image obtained is called D.
   e) Describe a single transformation that would map C onto D.
6. The coordinates of the vertices of quadrilateral T are (2, 2), (4, 2), (3, 3) and (2, 3).
   A is the transformation "rotate through $^-90$º about the centre (1, 0)".
   B is the transformation "translate by $\begin{pmatrix} ^-4 \\ 2 \end{pmatrix}$".
   a) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **BA**(T).
   b) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **AB**(T).
   c) Compare the positions of the final shape in a and b. Does the order matter?
7. The coordinates of the vertices of quadrilateral T are (2, 2), (4, 2), (3, 3) and (2, 3).
   A is the transformation rotate through $^-90$º about the centre (2, 1).
   B is the transformation rotate through 180º about the centre (2, 1).
   a) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **BA**(T).
   b) Draw T on square grid paper and find the single transformation equivalent to the combined transformation **AB**(T).
   c) Compare the positions of the final shape in a and b. Does the order matter?

Check your answers at the end of this unit.

Combining two or more transformations gives wide scope for investigative work. Combining translations might move into tessellations, another interesting area to investigate. As it is a practical activity involving making (and colouring) shapes that might give ‘attractive’ looking patterns for display, pupils most of the time enjoy this type of investigation and the starting point is such that all pupils, whatever their achievement level, can participate. The task is best done as group work, different groups working on different combinations of transformations. The final class product might be a display grid with the basic combinations illustrated for a shape S in each cell.

If T represents a translation, R($x = 0$) a reflection in the y-axis, R($y = 0$) a reflection in the x-axis, R($x = y$) and R($x = {}^-y$) reflections in the lines with equation $x = y$ and $x = {}^-y$ respectively and Rot($\theta°$) a rotation about O through $\theta°$, then the grid of combined transformations could look as illustrated below.

Reflections in vertical and/or horizontal lines in general ($x = a$, $y = b$) could be added to the investigations.

Further work can be done on considering more than two transformations.

	T	R ($x = 0$)	R ($y = 0$)	R ($x = y$)	R ($x = {}^-y$)	Rot (${}^+90°$)	Rot (${}^-90°$)	Rot ($180°$)
T								
R($x = 0$)								
R($y = 0$)								
R($x = y$)								
R($x = {}^-y$)								
Rot (${}^+90°$)								
Rot ($180°$)								
Rot (${}^-90°$)								

Section A2: Assessing investigative work

Assessment is part of the learning process and one of the most discussed (and at times controversial) issues. What is the purpose of assessment? How do we assess? What do we assess? are some of the questions that come to mind.

Write down your views and ideas on assessment. Consider questions such as:

What is assessment? Is assessment the same as testing and/or grading?

Why do you assess your pupils?

How do you assess your pupils?

How can you be sure that the task you set is assessing what you want to assess?

Do you assess pupils':

(1) ability to apply mathematical knowledge to solve problems in mathematics and other disciplines

(2) ability to use mathematical language to communicate ideas

(3) ability to reason and analyse using mathematical models/critical thinking

(4) knowledge and understanding of concepts and procedures

(5) disposition towards mathematics

(6) understanding of the nature of mathematics

If yes: how do you assess it? If no: why not?

Assessment of investigative work looks for different skills than other pieces of assessment. Assessment in general is the process of

planning the assessment: setting the objectives for the assessment. Answering the question: What is to be assessed?

gathering evidence about a pupil's (i) knowledge of, (ii) ability to use, and (iii) attitude towards mathematics and deciding what kind of task/activity will be most appropriate to assess the set objective.

interpreting the evidence gathered: What level of understanding is revealed by pupil's responses?

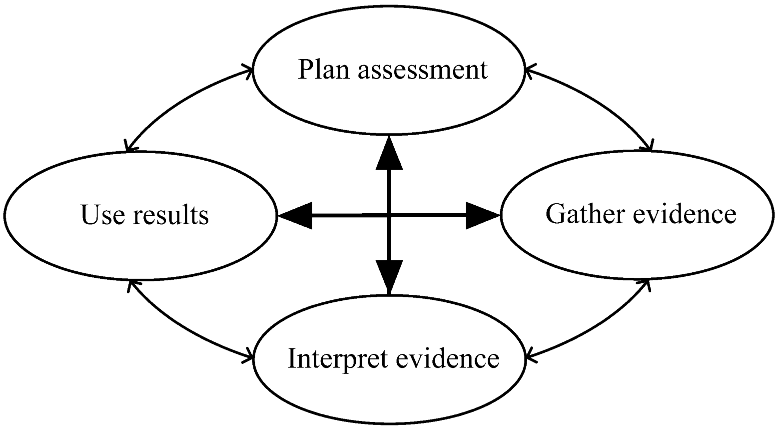
using this evidence for a variety of purposes:

for the pupil to enhance his/her learning

for the teacher to set appropriate learning activities: revision of teaching method, objectives set, decide on remedial activities (this cycles back into the first step)

for the society to ensure comparable standards/giving grades

The process is illustrated in the diagram below:



Assessment Phases

Assessment is a broad concept: to determine how far the educational aims and objectives have been achieved (by the pupils, the teacher, the learning aids, the teaching approach etc.) and should **not be equated to** quantitative measurement or **testing** (obtaining grades). Evaluation includes use of qualitative value judgment. For example assessment of communicative skills, practical skills, problem solving skills and personal qualities (flexibility, systematic working, independent thinking, cooperation, persistence on a task, interest in the subject, enjoying mathematics, confidence in own ability) will have to be based on observations and value judgments by the teacher.

Assessment is to form an integral part of teaching and learning. Assessment is to develop out of the curriculum, its aims and objectives. Pupils should learn authentic assessment to assist them in their learning process.

Among the many categorisations of assessment the distinction between continuous and discrete assessment is one worth looking at in some more detail.

Continuous assessment takes places concurrently with, and is integrated into, the teaching/learning process. It takes note and keeps record of the general progress of the individual pupil's performance, attainment against the set criterion, level of activity, working modes and attitudes, interest level, group participation—covering the full range of objectives stated for the learning of mathematics. Continuous assessment is assessment in context, within the day to day process of learning activities. Through continuous assessment a typical attainment level (pupils profile) of the pupils will emerge. Continuous assessment will assess both *product behaviour* (assessing the outcome or product of an activity—the worked out problem, a report, a model) and *process behaviour* (assessment of the skills used, the strategies employed, the attitudes displayed while completing a task).

Advantages of continuous assessment are:

1) it is more representative for the pupil's achievement across time and tasks
2) the pupil can demonstrate the achievement level over a longer period of time—it is not bound to the 'one-shot' occasion of a formal test
3) it will emphasise learning as a continuous process—pupils are to work continuously as they can be assessed at any time. This avoids "learning for the test/exam" the night before the test is to be taken
4) the pupil and teacher receive continuous feedback on the progress made
5) it can assess the whole range of objectives in the cognitive, affective and psycho motor domains

Discrete assessment takes place at a specific time (end of the course, specified time to hand in project or set assignment). Tests belong to the category of discrete assessment. They are generally conducted under standardized conditions (time constraints), set by the pupil's teacher or by teachers at the same institution. Test can be written, practical, aural-oral in nature and can take a variety of formats. When the assessment takes place at the end of a school year or course they are referred to as examinations. Examinations can be set i) internally (by teachers from the institution), ii) externally (by an external examination body) or iii) set internally with external moderation i.e. an external body checks whether the examination is of the required standard and whether the marking has been done fairly and consistently. Discrete assessment generally aims to provide data to make decisions on pass/fail, and to provide information to other educational institutions, employers and educational authorities.

It is not unusual for both forms of assessment to be used. From an educational point of view the continuous assessment mode is the most valuable and some will argue that decisions on pupils' futures should be taken based on continuous assessment only.

Investigations and problem solving assessment

Learning to solve problems is the **principal reason for studying mathematics.** Problem solving and investigative work is the process of applying previously acquired knowledge to new and unfamiliar situations. In the carrying out of an investigation and the reporting of the outcomes (orally or in writing) the following skills can be developed:

1) Communication skills
   Pupils explain, talk, discuss, question, agree, report.
2) Reasoning skills
   Pupils clarify, justify, conjecture, prove.
3) Operational skills
   Pupils collect data, sort, order information.
4) Recording skills
   Pupils draw, write, list, graph.

Investigative work and problem solving tasks are therefore to be an integral part of the learning of mathematics as it touches on the heart of mathematics.

To assess the four skills an assessment scheme has to be flexible to accommodate the wide range of possible responses of pupils and at the same time specific so different teachers will assess a piece of work of a particular pupil in the same way.

The following is a scheme that can be followed when assessing investigative and/or problem solving tasks. It looks at four categories: the overall design of the work and the strategy used, the mathematical content, the accuracy and the clarity of argument and presentation. In each category a score 0 - 4 can be awarded as specified below. It has been used by the University of Cambridge to assess coursework.

1) Overall design and strategy

Score	Criteria
0	Much help was received. No apparent attempt has been made to plan the work.
1	Help has been received from teacher and/or peers. Little independent work has been done. Some attempt to solve the problem but at a simple level. The work is poorly organized, showing little overall planning.
2	Some help has been received from the teacher or the peer group. A strategy has been outlined and an attempt made to follow it. A routine approach with little evidence of the pupil's own ideas being used.
3	The work has been satisfactorily carried out, with some evidence of forward planning. Appropriate techniques have been used, although some of these have been suggested by others, yet the development and use of them is the pupil's own.
4	The work is well planned and organized. The pupil has worked independently, devising and using techniques appropriate to the task. The pupil is aware of the wider implications of the task and has attempted to extend it. The outcome of the task is clearly explained.

2) Mathematical content

Score	Criteria
0	Little or no evidence of any mathematical activity. The work is very largely descriptive or pictorial.
1	A few concepts and methods relevant to the task have been used, but in a superficial and repetitive manner.
2	A sufficient range of mathematical concepts which meet the basic needs of the task have been employed. More advanced mathematical methods may have been attempted but not necessarily appropriately or successfully.
3	The concepts and methods usually associated with the task have been used and the pupil has shown competence in using them.

4 The pupil has used a wide range of mathematics competently and relevantly plus some beyond the usual and obvious. Some mathematical originality has been shown.

3) Accuracy

The mark for accuracy should normally not exceed the mark for mathematical content.

Score	Criteria
0	Very few calculations have been carried out and errors have been made in these. Diagrams and tables are poor and mostly inaccurate.
1	Correct work on limited mathematical content or calculations performed on a wider range with some errors. Diagrams and tables are adequate but units are often omitted or incorrect.
2	Calculations have been performed on all the topics relevant to the task with only occasional slips. Diagrams are neat and accurate, but routine; and tables contain information with a few errors. The pupil has shown some idea of the appropriate degree of accuracy for the data used. Units are used correctly.
3	All the measurements and calculations associated with the task have been completed accurately. The pupil showed an understanding of magnitude and degree of accuracy when making measurements or performing calculations. Accurate diagrams are included which support the written work.
4	Careful, accurate and relevant work throughout. This includes, where appropriate, computation, manipulation, construction and measurements with correct units. Accurate diagrams are included which positively enhance the work, and support the development of the argument. The degree of accuracy is always correct and appropriate.

4) Clarity of argument and presentation

Score	Criteria
0	Haphazard organization of work which is difficult to follow. A series of disconnected short pieces of work. Little or no attempt to summarize the results.
1	Poorly presented work, lacking logical development. Undue emphasis is given to minor aspects of the task, whilst important aspects are not given adequate attention. The work is presented in the order in which it happened to be completed; no attempt is made to re-organize it into a logical order.
2	Adequate presentation which can be followed with some effort. A reasonable summary of the work completed is given, though with some lack of clarity and/or faults of emphasis. The pupil has made some attempt to organize the work in logical order.

3 A satisfactory standard presentation has been achieved. The work has been arranged in logical order. Adequate justification has been given for any generalization made. The summary is clear, but the pupil has found some difficulty in linking the various different parts of the task together.

4 The presentation is clear, using written, diagrammatic and graphical methods as and when appropriate. Conclusions and generalizations are supported by reasoned statements which refer back to results obtained in the main body of the work. Disparate parts of the task have been brought together in a competent summary. The summary refers to the aims. There are good suggestions to extend the task or apply in other areas.

Unit 4, Practice activity

1. The combination of transformations is a rich topic for pupils to investigate. Set an investigative activity for your pupils on the combination of two transformations. Give different tasks to groups of four pupils and have them present their findings (including a poster display) to the whole class.

   Use the given assessment scheme to assess pupils' work.

   Evaluate the activity. Attach the tasks set and (samples of) pupils' work.

Present your assignment to your supervisor or study group for discussion.

Section B: Transformations and matrices

In this section you will describe transformations using matrices. A matrix is a convenient mathematical tool for representing transformations of coordinates. It is assumed that you have basic knowledge of matrices and operations with matrices. The basic concepts will shortly be recalled in the next section.

> Write down what you recall about matrices. Some questions to reflect on could be:
> What is a matrix? Where can they be used? What real life situations can be represented using matrices? What operations with matrices can you recall? Do you cover matrices with your pupils? If yes, how do introduce matrices? What context do you use?

Section B1: Matrices and operations with matrices

Information technology (IT) is about storing, analysing and retrieving information by computer. A lot of data are presented in rectangular array format or matrix format.

For example
The results (lose L, win W or draw S) of a football tournament between five teams, P, Q, R , S and T are represented in the matrix

$$\begin{array}{c} \\ \\ \text{Team} \\ \\ \\ \end{array}\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array}\begin{array}{c} \quad\text{Result} \\ \begin{array}{ccc} L & W & D \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \end{array}$$

The matrix has 4 rows and 3 columns. The **order** is said to be 4 by 3. The convention is to name the number of rows first followed by the number of columns.

The order of a matrix (number of rows) by (number of columns).

The matrix can be described in words as a team by result matrix.
The first row carries the information that team P lost 1 game, won 2 and drew 1.

The sum of the entries in the first row 1 + 2 = 1 = 4 tell you that team P played 4 matches altogether.

The sum of the entries in the first column 1 + 2 + 1 + 2 + 1 = 7 tells you that of all the matches played 7 ended in a loss for one of the teams (hence a win for the other, so the total of the W column must be the same! Check this.).

A **square matrix** is a matrix with the number of rows equal to the number of columns.

A wide variety of data can be displayed in matrix format as you will see in the next self mark exercise. And more information than just the data appearing can be obtained from the matrix (sum of row/column totals for example frequently give information).

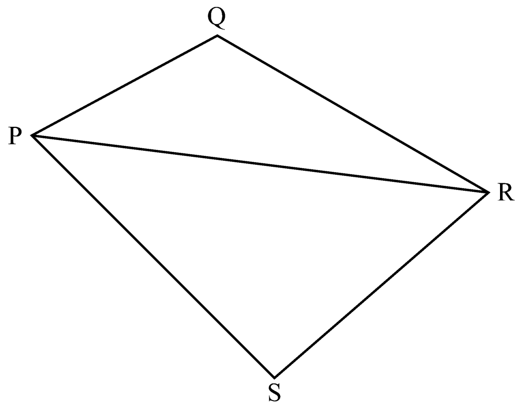
Self mark exercise 2

1. A factory produces T-shirts in the sizes small (S), medium (M), large (L) and extra large (XL). To make a T-shirt ready for distribution to wholesalers time is needed for cutting (C), for sewing (S) and for packing (P). The matrix displays information on the number of minutes of each production activity for each type of T-shirt.

$$\begin{array}{cc} & \text{time} \\ & \begin{array}{ccc} C & S & P \end{array} \\ \text{T-shirt} \begin{array}{c} S \\ M \\ L \\ XL \end{array} & \begin{pmatrix} 10 & 20 & 5 \\ 12 & 24 & 5 \\ 15 & 28 & 5 \\ 15 & 30 & 5 \end{pmatrix} \end{array}$$

   a) How many minutes does it take (on average) to cut a medium sized T-shirt?
   b) How many minutes does it take to sew an extra large T-shirt?
   c) What is the total time to make and pack a large T-shirt?
   d) What is the order of the matrix?
   e) Describe the matrix using words.
   f) What information is given by the sum of the times in a row?
   g) What information is given by the sum of the times in a column?

2. The diagram illustrates the roads between the places P, Q, R and S.



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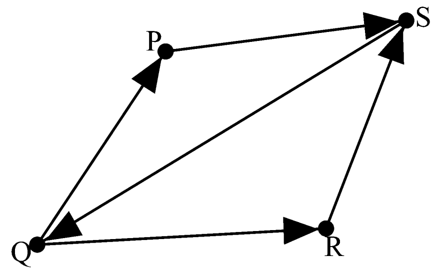
This information can be stored in a matrix. 0 indicates NO road between the two places. 1 indicates that there is one road. The first row has been completed.

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{From} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}$$

a) Explain the entries in the first row.
b) Complete the matrix.
c) What meaning can you give the total of each row?
d) What meaning can you give to the total of each column?
e) What is the order of the matrix?
f) Why are there zeros on the (main) diagonal?

3. The following diagram gives a directed route map.



a) Complete the directed route matrix

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{From} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}$$

b) Find the total of each row and column and interpret the values.

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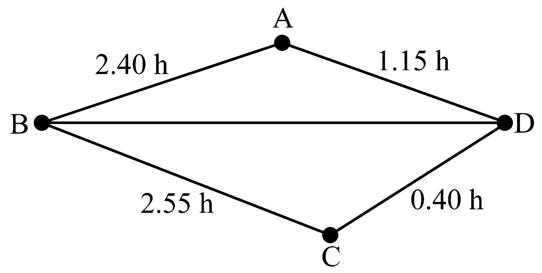
4. The diagram illustrates times needed to travel between places.



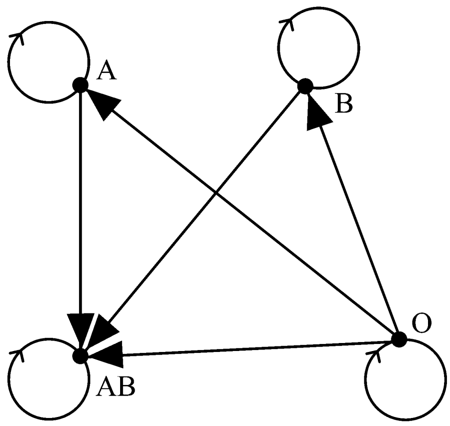
   a) Represent the data in a matrix.

   b) What is the shortest time to travel from B to D? What route do you have to take in that case?

   c) In the matrix each number appears twice. Explain why this is so.

   d) When would a travel time matrix not contain twice the same entries?

5. Blood of one person cannot just be given to another person. The blood types that are generally considered are A, B, AB and O. The directed graph illustrates who can donate blood to whom.



   a) To people of which blood group can persons with blood group AB donate blood?

   b) From which blood groups can people with blood group AB receive blood?

   c) Person with blood group O are sometimes described as 'universal donors'. Can you explain?

   d) You are to go on a dangerous expedition with a group of 10 people with unknown blood groups. Which blood type would you take along for emergency situations and why?

   e) Represent the data of the graph in a matrix.

   f) What is the meaning of the total of the entries in each row?

   g) What is the meaning of the total of the entries in each column?

Check your answers at the end of this unit.

Section B2: Operations with matrices

Matrices can be added, subtracted, multiplied by a constant (scalar multiplication) or multiplied with each other. In the following excercise you will look at some situations requiring these operations.

Self mark exercise 3

1. Three sales points for cars, one each in Ftown, Gtown and Htown, sell a standard model S and a deluxe model L. The sales for the first week of August are given in the matrix A.

$$\text{Garage}\quad \begin{array}{c} \\ F \\ G \\ H \end{array}\!\!\begin{array}{c} \text{Model} \\ \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 6 & 2 \\ (2 & 0) \\ 4 & 1 \end{pmatrix} \end{array} = A.$$

The sales for the second week of the month are given by the matrix B.

$$\text{Garage}\quad \begin{array}{c} \\ F \\ G \\ H \end{array}\!\!\begin{array}{c} \text{Model} \\ \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 4 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \end{array} = B.$$

a) What is the meaning of each row total?

b) What is the meaning of each column total?

c) Express in a matrix the total sales of each store over the two week period.

d) What is the order of the sum matrix?

e) What information do the row totals and column totals of the sum matrix give you?

f) To add two matrices M and N there has to be something special about their orders. What?

2. Three clothing shops A, B and C sells three types of jeans. The number of each type they have in stock is given by the stock matrix M.

$$\text{Stock}\quad \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \text{type of jeans} \\ \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 800 & 900 & 1000 \\ 600 & 600 & 1200 \\ 600 & 750 & 950 \end{pmatrix} \end{array} = M$$

The sales of each type by each store during a month are represented in matrix N.

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$$\begin{array}{cc} & \text{type of jeans} \\ & \begin{array}{ccc} R & S & T \end{array} \\ \text{Stock} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 800 & 600 & 1000 \\ (600 & 200 & 1100) \\ 580 & 550 & 940 \end{pmatrix} = N \end{array}$$

a) What information is given by the row totals of matrix M?

b) What information is given by the column totals of matrix M?

c) Write down the matrix representing the stock in each store by the end of the month.

d) Which type of jeans sold best?

e) Which type of jeans is unpopular?

3. A bakery makes two types of cakes A and B. The main ingredients, in grams, needed for each type are indicated in the following matrix.

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array} & \begin{pmatrix} 250 & 200 \\ 120 & 150 \\ 100 & 125 \end{pmatrix} = I \end{array}$$

a) What does the row total tell you?

b) What information is given by column totals?

c) If the bakery wants to make 8 of each what amount of ingredients is needed for each type? Write the amounts in a matrix.

Check your answers at the end of this unit.

In the above exercise you reviewed, in context, the addition, subtraction and scalar multiplication of matrices. The following was recalled:

(1) Matrices of the same order can be added, or subtracted, by adding or subtracting corresponding entries. For example for 2 by 2 matrices you have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a-p & b-q \\ c-r & d-s \end{pmatrix}$$

(2) Matrices can be multiplied by a number (scalar) by multiplying each entry by the number.

For example if a 2×2 matrix is multiplied by the constant k:

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

A n by m matrix can be in general represented by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots\dots\dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots\dots\dots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \dots\dots\dots & a_{3m} \\ .. & .. & .. & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{pmatrix}$$

The entry in the ith row and jth column is denoted by a_{ij}

The general form is useful to refer to entries in the matrix e.g. a_{34} is the entry at the intersection of the third row and the fourth column (or the fourth element in the third row; or the third element in the fourth column).

The multiplication of two matrices is based on a **row** × **column** technique as illustrated in the following example.

Miss Mpete owns two stores A and B in different parts of the town. The stores sell the same items at the same price. Tomato paste is available in each of the stores in bottles of two different sizes (small and large). The following stock matrix S displays the number of bottles of each type in each of the stores.

$$\mathbf{S} = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \text{Small} \quad \text{Large} \\ \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix} \end{array}$$

The matrix P gives information on the cost price (in P) of each bottle.

$$\mathbf{P} = \begin{array}{c} \\ \text{Small} \\ \text{Large} \end{array} \begin{array}{c} \text{CP} \\ \begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix} \end{array}$$

What is the value of the stock in each shop?

In store A the value of the stock is 40 × P 3.20 + 20 × P 4.60 = P 220.-

In store B the value of the stock is 30 × P 3.30 + 10 × P 4.60 = P 142.-

This can be represented as the product of the two matrices **SP**

$$\mathbf{SP} = \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix} \begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix} = \begin{pmatrix} 40 \times 3.20 + 20 \times 4.60 \\ 30 \times 3.20 + 10 \times 4.60 \end{pmatrix} = \begin{pmatrix} 220 \\ 142 \end{pmatrix}$$

The (store × **size**) matrix has been multiplied by the (**size** × cost price) matrix leading to a (store × stock value) matrix. The two matrices are compatible: the number of columns in the first is equal to the number of rows in the second.

In store A the value of the stock is P 220.- and in the store B the value is P 142.

Note that the entries of the first row (a_{11} and a_{12}) have been multiplied by the entries in the first column (b_{11} and b_{21}) and added ($p_{11} = a_{11}b_{11} + a_{12}b_{21}$) to obtain the first entry in the product matrix.

$$\begin{pmatrix} 40 & 20 \\ .. & .. \end{pmatrix}\begin{pmatrix} 3.20 \\ 4.60 \end{pmatrix}$$

Similarly to obtain the entry in the product matrix p_{21} (second row first element) the entries of the second row of S are multiplied by the entries in the first column of **P** ($p_{21} = a_{21}b_{11} + a_{22}b_{21}$)

In general for the product of a 2×2 matrix and a 2×1 matrix we have:

$$\mathbf{AX} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

For the multiplication of two matrices **M** and **N** to be possible the number of columns in the first (say M) is to be equal to the number of rows in the second matrix (N).

If **M** is $n \times m$ and **N** is $p \times q$ the product **MN** can only be formed if $m = p$. The product matrix will have order $n \times q$. (Check this statement with some examples.)

$(n \times m) \quad (p \times q)$

SAME

order of
Product Matrix

Here is another example on the use of matrix multiplication.

Godirwang, Moreti and Mashaka obtained from the school supplies offices the items shown in the table below for the first term.

	Pencils (Pc)	Pens (Pe)	Exercise books (Ex)
Godirwang	2	3	6
Moreti	1	4	5
Mashaka	2	2	7

Representing the data in the table in a (student ($n \times m$) item) matrix:

$$\mathbf{M} = \begin{array}{l} \\ \text{Godirwang} \\ \text{Moreti} \\ \text{Mashaka} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Pc} & \text{Pe} & \text{Ex} \end{array} \\ \begin{pmatrix} 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 2 & 7 \end{pmatrix} \end{array}$$

If pencils sell at 65t, pens at 85t and exercise books at 150t these data can be represented in an (item $\times$ cost) matrix **C:**

$$\mathbf{C} = \begin{array}{l} \\ \text{Pc} \\ \text{Pe} \\ \text{Ex} \end{array} \begin{array}{c} \text{Cost} \\ \begin{pmatrix} 65 \\ 85 \\ 150 \end{pmatrix} \end{array}$$

The matrix product **MC** is the product of the (student × item) matrix and the (item × cost) matrix, which will give a (student × total cost) matrix.

$$\mathbf{MC} = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 4 & 5 \\ 2 & 2 & 7 \end{pmatrix}\begin{pmatrix} 65 \\ 85 \\ 150 \end{pmatrix} = \begin{pmatrix} 2\times 65 + 3\times 85 + 6\times 150 \\ 1\times 65 + 4\times 85 + 5\times 150 \\ 2\times 65 + 2\times 85 + 7\times 150 \end{pmatrix} = \begin{pmatrix} 1285 \\ 1155 \\ 1350 \end{pmatrix}$$

Godirwang has to pay P 12.85, Moreti P 11.55 and Mashaka P 13.50

The above examples multiplied a matrix with a column matrix to make the row column multiplication clear. The process applies to any two compatible matrices.

For example the store matrix from an earlier example could be combined not only with a matrix giving the cost price but the matrix could also include the profit.

The stock matrix **S** giving the number of bottles of each type of tomato in each of the stores was

$$\mathbf{S} = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} \text{Small} & \text{Large} \end{array} \\ \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix} \end{array}$$

The matrix P gives information on the cost price (in P) of each bottle and the profit

$$\mathbf{P} = \begin{array}{c} \\ \text{Small} \\ \text{Large} \end{array}\begin{array}{c} \begin{array}{cc} \text{CP} & \text{Profit} \end{array} \\ \begin{pmatrix} 3.20 & 0.30 \\ 4.60 & 0.42 \end{pmatrix} \end{array}$$

The product of the two matrices **SP** now gives

$$\begin{aligned} \mathbf{SP} &= \begin{pmatrix} 40 & 20 \\ 30 & 10 \end{pmatrix}\begin{pmatrix} 3.20 & 0.30 \\ 4.60 & 0.42 \end{pmatrix} \\ &= \begin{pmatrix} 40\times 3.20 + 20\times 4.60 & 40\times 0.30 + 20\times 0.42 \\ 30\times 3.20 + 10\times 4.60 & 30\times 0.30 + 10\times 0.42 \end{pmatrix} \\ &= \begin{pmatrix} 220 & 20.40 \\ 142 & 13.20 \end{pmatrix} \end{aligned}$$

The product matrix tells you that in store A the stock has a value of P 220 and if completely sold a profit of P 20.40 will be made. For store B the figures are P 142 for stock value and P 13.20 for profit if all sold.

In general form the product of two 2 × 2 matrices can be expressed as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

Self mark exercise 4

1. Two factories A and B employ both skilled and unskilled workers. The data is given by the employment matrix E. The average monthly wage paid, in Pula, to each category of worker is represented in matrix W.

$$E = \begin{array}{c} \\ \text{Factory A} \\ \text{Factory B} \end{array} \begin{array}{c} \begin{array}{cc} \text{Skilled} & \text{Unskilled} \end{array} \\ \begin{pmatrix} 40 & 90 \\ 30 & 50 \end{pmatrix} \end{array} \qquad W = \begin{array}{c} \\ \text{Skilled} \\ \text{Unskilled} \end{array} \begin{array}{c} \text{Wage} \\ \begin{pmatrix} 650 \\ 420 \end{pmatrix} \end{array}$$

   a) Find the product EW to find the monthly wage bill for each factory.
   b) Can you find WE?

2. Matrix A has order $m \times n$ and matrix B has order $p \times q$.
   a) If BA exists what can you say about m, n, p, q?
   b) What is the order of BA?
   c) If A^2 exists, that is AA, what is special about A?

3. The matrix shows the results for a five team tournament.

$$\text{Team} \begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccc} L & W & D \end{array} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \end{array} = M$$

   The points given are 2 for a win, 1 for a draw and 0 for a loss and can be represented in the matrix N.

$$\begin{array}{c} \text{lose} \\ \text{win} \\ \text{draw} \end{array} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = N$$

   Find the product matrix MN and interpret the entries.

4. A factory produces T-shirts in the sizes small (S), medium (M), large (L) and extra large (XL). To make a T-shirt ready for distribution to wholesalers time is needed for cutting (C), for sewing (S) and for packing (P). The matrix displays information on the time in hours of each production activity for each type of T-shirt.

$$\text{T-shirt} \begin{array}{c} \\ \\ S \\ M \\ L \\ XL \end{array} \begin{array}{c} \text{time} \\ \begin{array}{ccc} C & S & P \end{array} \\ \begin{pmatrix} 0.2 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.25 & 0.45 & 0.1 \\ 0.25 & 0.5 & 0.1 \end{pmatrix} \end{array} = A$$

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The matrix O gives information on the order for the next months June and July.

$$O = \begin{array}{c} \\ \text{June} \\ \text{July} \end{array} \begin{array}{c} \begin{array}{cccc} S & M & L & XL \end{array} \\ \begin{pmatrix} 600 & 750 & 1000 & 300 \\ 550 & 800 & 1100 & 300 \end{pmatrix} \end{array}$$

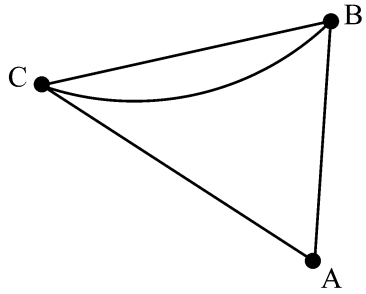
a) Find OA and interpret the result.

b) What is the total sewing time (in days and hours) needed to prepare the June order?

c) What is the total time needed for packing of the July order?

d) Is it possible to compute AO? Explain.

5. The diagram show the direct roads linking three town A, B and C,



a) Copy and complete the directed route matrix R.

$$\text{From} \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \text{to} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \end{pmatrix} \end{array} = R$$

b) Interpret the meaning of the total of each row and of the total of each column.

c) Work out the matrix product R R = R^2

d) Interpret the entries in the matrix R^2, as well as meaning of row totals and column totals.

e) R^2 is called the two-stage route matrix. Explain.

6. Given the matrices I and A

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$A^2 = A \cdot A$, $A^3 = A \cdot A^2$, and so on.

a) Given that $A^2 = pA + qI$, show that $p = 4$ and $q = {}^-1$

b) Show that $A^3 = 15A - 4I$

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7. If A and O (zero matrix) are the matrices

   $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A^2 = O$ find, assuming no entry is 0,

   a) the relationship between a and d

   b) the relationship between a^2 and bc

   c) all the matrices A with $A^2 = O$, $a = 6$ and b, c integers

8. If A, B and C are three 2×2 matrices show that in general

   a) $AB \neq BA$ (matrix multiplication is NOT commutative)

   b) $(AB)C = A(BC)$ (matrix multiplication is associative)

Check your answers at the end of this unit.

Section B3: Matrices to represent transformations

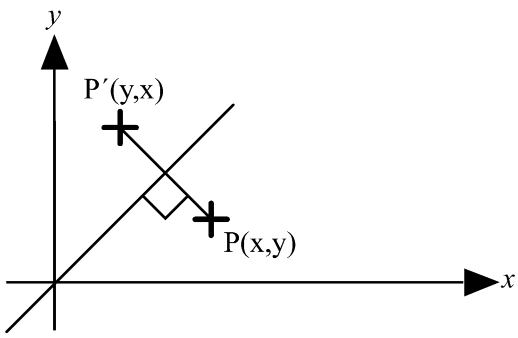
The previous section was to recall matrix operations. Matrices are convenient algebraic tools to use for representing transformation.

The linear transformation that maps the point with coordinates (x, y) onto the point with coordinates (x', y') where $x' = ax + by$, $y' = cx + dy$ can be represented as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 1

If $P(x, y)$ is reflected in the line with equation $x = y$ the image P' has coordinates (y, x).



$P(x, y) \Rightarrow P'(x', y')$ where $x' = y$ and $y' = x$.

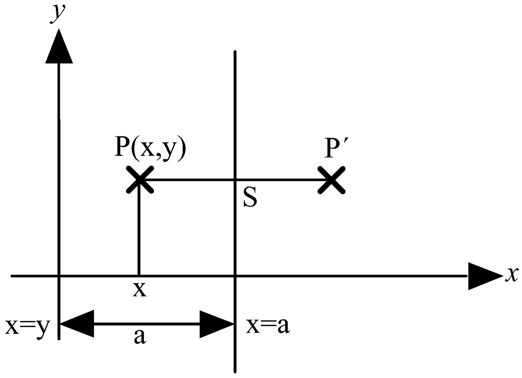
Expressed with matrices:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0x + 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection in the line with equation $x = y$.

Example 2

The point P(x, y) is reflected in the line with equation $x = a$



PS = $a - x$ and SP′ = $a - x$. The x-coordinate of P′ is therefore $x + 2(a - x) = 2a - x$.

The coordinates of the image P′ are ($2a - x$, y).

In matrix form this becomes

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2a - x \\ y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^-x \\ y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^-x + 0y \\ 0x + 1y \end{pmatrix} = \begin{pmatrix} 2a \\ 0 \end{pmatrix} + \begin{pmatrix} {}^-1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} {}^-1 & 0 \\ 0 & 1 \end{pmatrix}$ represents a reflection in the y-axis. The format expresses that to obtain P′ one could first reflect P in the y-axis and shift the image obtained by $\begin{pmatrix} 2a \\ 0 \end{pmatrix}$

In the above two examples, a known transformation is described using matrices. The other way around is also possible. If you are given a matrix to transform a shape S onto S′one can (in some cases) describe the transformation S $\Rightarrow$ S′ in terms of known transformations. This is illustrated in the following example.

Example 3

Transform triangle ABC with coordinates of the vertices A(1, 2), B(2, 0) and C(4, 3) by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

The coordinates of the given triangle can be represented in a matrix

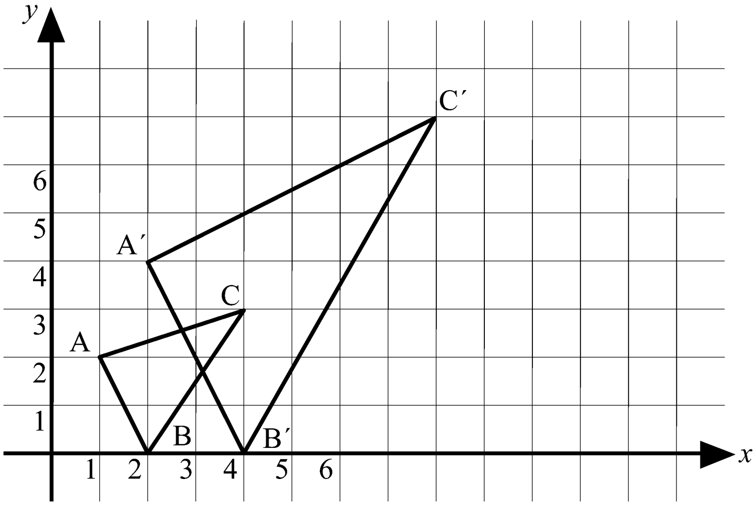
$$\mathbf{P} = \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \end{pmatrix} \end{array}$$

The transformation can now be carried out by computing the matrix product MP

$$\mathbf{MP} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathrm{A}' & \mathrm{B}' & \mathrm{C}' \end{array} \\ \begin{pmatrix} 2 & 4 & 8 \\ 4 & 0 & 6 \end{pmatrix} \end{array}$$

The image triangle A′B′C′ has as coordinates of its vertices A′(2, 4), B′(4, 0) and C(8, 6).

The diagram illustrates the triangle ABC and its image A′B′C′ under the transformation given by the matrix M.



Triangle A′B′C′ is an enlargement from the triangle ABC with centre O and scale factor 2.

The matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement centre O, scale factor 2.

Self mark exercise 5

1. The coordinates of the vertices of ΔPQR are P(1, 3), Q(2, $^{-}3$) and R($^{-}3$, $^{-}1$).

   a) Represent the coordinates of ΔPQR in a matrix A.

   b) $\mathbf{M} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is a transformation matrix applied to the vertices of ΔPQR to give the image ΔP′Q′R′.
   Plot the points P′, Q′ and R′ and draw ΔP′Q′R′ for $k = {}^{-}2$. Describe the transformation that maps ΔPQR onto ΔP′Q′R′.

   c) Investigate the transformation given by MA taking all kinds of values for k.

2. If the matrix $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ {}^{-}1 & 0 \end{pmatrix}$, calculate and interpret the product RA by plotting ΔPQR and its image ΔP′Q′R′ after applying matrix R.

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3. Find the image of ΔPQR with vertices P(0, 0), Q(2, 1) and R(1, 3) under each of the following transformations. In each case describe the transformation in words and represent the transformation in matrix form.

   a) (x, y) maps onto $(x, {}^-y)$ b) (x, y) maps onto $(y, {}^-x)$

   c) (x, y) maps onto $(0, y)$ d) (x, y) maps onto $(x, 6\ {}^-\ y)$

4. Find the image of ΔABC with vertices A(0, 0), B(2, 3) and C(1, 4) under each of the following matrix transformations. Describe in words the transformation in each case.

   a) $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ b) $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ c) $\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ d) $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

5. Investigate and describe the transformation

   $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} + \begin{pmatrix} 0 & {}^-1 \\ {}^-1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Check your answers at the end of this unit.

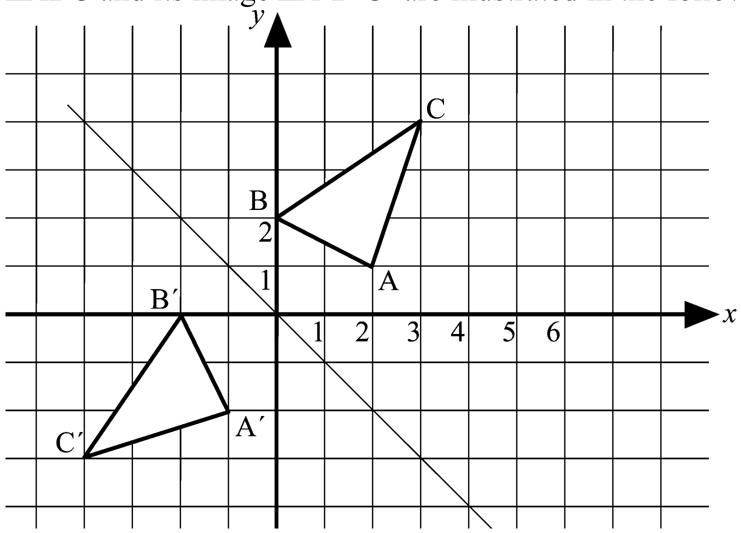
At the start of this unit you studied combined transformations: the effect of applying a transformation **A** on a shape S to give S′ and then continuing to apply a transformation **B** to S′ to give as image S″. It was possible in several cases to describe a single transformation that would map S onto S″. Now that you have studied how to represent transformation by matrices, a combined transformation BA(S) will be described by the matrix product of **B** and **A**. This is illustrated in the following example:

The ΔABC with vertices A(2, 1), B(0, 2) and C(3, 4) is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 0 & {}^-1 \\ {}^-1 & 0 \end{pmatrix}$

Do you recognise the matrix? What transformation is it representing?
The vertices of the image ΔA′B′C′ are obtained by the matrix multiplication:

$$\mathbf{MP} = \begin{pmatrix} 0 & {}^-1 \\ {}^-1 & 0 \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} {}^-1 & {}^-2 & {}^-4 \\ {}^-2 & 0 & {}^-3 \end{pmatrix} \end{array}$$

ΔABC and its image ΔA´B´C´ are illustrated in the following diagram.



The triangles are each others reflection in the line with equation $x = y$.

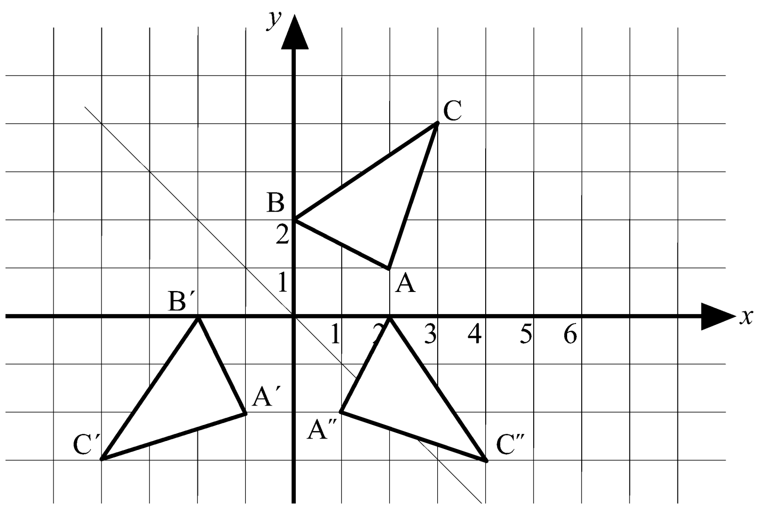
To ΔA´B´C´ the transformation given by the matrix $\mathbf{N} = \begin{pmatrix} {}^{-}1 & 0 \\ 0 & 1 \end{pmatrix}$ is applied.

The coordinates of the image ΔA″B″C″ are calculated as follows:

$$\mathbf{N(MP)} = \begin{pmatrix} {}^{-}1 & 0 \\ 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathrm{A}' & \mathrm{B}' & \mathrm{C}' \end{array} \\ \begin{pmatrix} {}^{-}1 & {}^{-}2 & {}^{-}4 \\ {}^{-}2 & 0 & {}^{-}3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathrm{A}'' & \mathrm{B}'' & \mathrm{C}'' \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ {}^{-}2 & 0 & {}^{-}3 \end{pmatrix} \end{array}$$

ΔA″B″C″ is a reflection of triangle ΔA´B´C´ in the y-axis.

Illustrating ΔA″B″C″ in the diagram gives:



The transformation which maps ΔABC onto $\Delta A''B''C''$ can be recognised from the diagram as a rotation about O through $^{-}90^{\circ}$. This is confirmed by the matrix product NM.

$$\mathbf{NM} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The product matrix representing a rotation about O through $^{-}90^{\circ}$

Applying the product matrix to the original ΔABC should give $\Delta A''B''C''$.

$$\mathbf{(NM)P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 0 & 3 \\ 1 & 2 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A'' & B'' & C'' \end{array} \\ \begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & -3 \end{pmatrix} \end{array}$$

In words: a reflection in the line with equation $x = y$ followed by a reflection in the y-axis is equivalent with a rotation about O through $^{-}90°$.

In matrix notation we could write $R_{x=0}\, R_{x = y} = \text{Rot-}_{90}$

The product $\mathbf{MN} = \begin{pmatrix} 0 & ^{-}1 \\ ^{-}1 & 0 \end{pmatrix} \begin{pmatrix} ^{-}1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & ^{-}1 \\ 1 & 0 \end{pmatrix}$ representing a reflection in the line with equation $x = 0$ followed by a reflection in the line with equation $x = y$ gives a different combined transformation (rotation about O through 90°). This illustrates that matrix multiplication is (in general) not commutative. In your assignment for this unit you are asked to investigate the various combinations of transformations given in matrix form.

Unit 4, Investigation

1. a) Represent each of the following basic transformations in matrix form.

   R_x denotes a reflection in the x-axis

   R_y denotes a reflection in the y-axis

   $R_{x=a}$ denotes a reflection in the line with equation $x = a$

   $R_{y=b}$ denotes a reflection in the line with equation $y = b$

   $R_{x=y}$ denotes a reflection in the line with equation $x = y$

   $R_{x=^-y}$ denotes a reflection in the line with equation $x = {}^-y$

   $R_{^-90}$ denotes a rotation about the origin through $^-90°$

   R_{90} denotes a rotation about the origin through 90°

   R_{180} denotes a rotation about the origin through 180°

   E_k denotes an enlargement by scale factor k, centre (0, 0)

   b) Systematically investigate the combination of any two of the above transformations using the matrix format. Include also the translation $T\begin{pmatrix} a \\ b \end{pmatrix}$ (Do not forget to look at the same type of transformation applied twice, for example a reflection in a vertical line followed by a reflection in another vertical line or two translations following each other.)

   c) Use the assessment scheme for investigative work to assess your own work.

2. As an extension to the above consider

   (i) product of more than two matrices

   (ii) to include stretches, shears, rotations about O through any angle of size θ°, reflections in a general line with equation $y = mx + n$

   (iii) investigate transformation in 3D

Present your assignment to your supervisor or study group for discussion.

Module 3, Practice activity

1. a) Coordinate geometry can be used to prove geometrical facts that are, at the secondary school level, most of the time derived using symmetry, similarity or congruence properties.

   Design a worksheet for your pupils to prove using coordinate geometry some properties of quadrilaterals, given only the defining property of the quadrilateral. For example:

   (i) In a parallelogram the diagonals are equal in length (a parallelogram is a quadrilateral with two pairs of opposite sides parallel)

   (ii) In a parallelogram the diagonals bisect each other

   (iii) In a rhombus the diagonals are perpendicular to each other (a rhombus is a parallelogram with equal sides)

   (iv) If ABCD is a quadrilateral and P, Q, R and S are the midpoints of the sides AB, BC, CD and AD respectively, PQRS is a parallelogram

   b) Try out the worksheet with your pupils and write an evaluative report.

2. Defend with sound educational arguments the statement “ Decisions on pupils future should be taken based on continuous assessment in the classroom only.”

3. a. Design an investigation for pupils involving matrices in a realistic context (directed route matrices for example).

   b. Set the investigations to your pupils and mark their work using the scheme included in this module. Write an evaluative report, including some (samples) of pupil’s work to support your evaluation.

Present your assignment to your supervisor or study group for discussion.

Summary

You have come to the end of this module on analytical and transformation geometry. It is expected that you have reviewed and increased your knowledge on coordinate and transformation geometry, and their connection to geometry and algebra.

Apart from having increased your own knowledge you should have practiced teaching methods that might not have been part of your practice before you started with this module. The experience in the classroom with a pupil centred approach using, among others, games, challenging questions and problem solving/investigation activities should have widened your classroom practice and methods. The task of a teacher is in the first place to create an environment for the pupils in which they can learn by doing mathematics. Your final module assignment is to assess the progress you have made.

Answers to self mark exercises

Self mark exercise 1

1. a) translation by $\begin{pmatrix} ^{-}4 \\ 0 \end{pmatrix}$ b) translation by $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

   c) BA(S) $\neq$ AB(S), not commutative

2. ab) reflection in y-axis ($x = 0$) c) BA(S) = AB(S), commutative

3. ab) rotation about O through $^{+}90^{o}$ c) BA(S) = AB(S), commutative

4. e) rotation about O through $^{+}90^{o}$

5. e) reflection in the line with equation $x = 1$

6. a) rotation about (0, 3) through $^{-}90^{o}$
   b) rotation about (4, 1) through $^{-}90^{o}$
   c) BA(S) $\neq$ AB(S), not commutative

7. ab) rotation about (2, 1) through 90^{o}

Self mark exercise 2

1. a) 12 min b) 30 min c) 48 min d) 4×3
   e) type of T-shirt by production time matrix
   f) total time to make a particular type of T-shirt
   g) total time for cutting, sewing and packing one T-shirt of each type

2. a) from P there are roads to Q, to R and to S

   b)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{from} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

   c) total number of different roads by which you can leave the place

   d) total number of different roads by which you can enter a place

   e) 4×4

   f) no (circular) road from a place to itself

3\. a)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{from} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

b) row totals give the number of different roads by which you can leave the place

column totals give the total number of different roads by which you can enter a place

4\. a)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ \text{from} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} 0 & 2.40 & 0 & 1.15 \\ 2.40 & 0 & 2.55 & 3.50 \\ 0 & 2.55 & 0 & 0.40 \\ 1.15 & 3.50 & 0.40 & 0 \end{pmatrix} \end{array}$$

b) 3.35 h, travelling B $\Rightarrow$ C $\Rightarrow$ D

c) time from e.g. A to B is assumed to be the same as time taken from B to A.

d) if time to travel from e.g A to B would be different from the time to travel from B to A (road from A to B could be mainly ‘down hill’ and from B to A ‘uphill and hence take longer)

5\. a) to AB only

b) from all groups

c) they can donate blood to persons with any blood group

d) O (universal donor)

e)

$$\begin{array}{cc} & \text{to} \\ & \begin{array}{cccc} A & B & O & AB \end{array} \\ \text{from} \begin{array}{c} A \\ B \\ O \\ AB \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

f) total number of blood groups, blood can be donated to by that blood group

g) total number of blood groups, blood can be received from by that blood group

Self mark exercise 3

1. a) total number of cars of all types sold in each garage

   b) total number of each type of car sold in the three garages together

   c)

Model

$$A + B = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 6+4 & 2+0 \\ 2+1 & 0+1 \\ 4+2 & 1+1 \end{pmatrix} \end{array} = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{cc} S & L \end{array} \\ \begin{pmatrix} 10 & 2 \\ 3 & 1 \\ 6 & 2 \end{pmatrix} \end{array}$$

   d) 3×2

   e) Row totals: total number of cars sold in a garage over the two week period

   Column totals: total number of cars from model S (or L) sold in the three garages together during the two week period.

   f) The matrices to be added need to be of the same order and give a sum matrix of the same order.

2. a) Total number of jeans in stock in each store irrespective of size

   b) Total number of jeans of a particular size available in the three stores together

   c)

Type of jeans

$$M - N = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 800-800 & 900-600 & 1000-1000 \\ 600-600 & 600-200 & 1200-1100 \\ 600-580 & 750-550 & 950-940 \end{pmatrix} \end{array} = \begin{array}{c} \\ F \\ G \\ H \end{array}\begin{array}{c} \begin{array}{ccc} R & S & T \end{array} \\ \begin{pmatrix} 0 & 300 & 0 \\ 0 & 400 & 100 \\ 20 & 200 & 10 \end{pmatrix} \end{array}$$

   d) Type R

   e) Type S

3. a) Total amount (in g) of each ingredient needed to make one cake of each type (flour 550 g, margarine 270 g and sugar 225 g).

   b) Total mass (in g) of basic ingredients in each type of cake (470 for type A, 575 g for type B).

   c)

$$8 \times I = \begin{array}{r} \\ \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 8 \times 250 & 8 \times 200 \\ 8 \times 120 & 8 \times 150 \\ 8 \times 100 & 8 \times 125 \end{pmatrix} \end{array} = \begin{array}{r} \\ \text{Flour} \\ \text{Margarine} \\ \text{Sugar} \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 2000 & 1600 \\ 960 & 1200 \\ 800 & 1000 \end{pmatrix} \end{array}$$

Self mark exercise 4

1. a) Factory A P 63 800, factory B P 40 500

   b) No the matrices are not compatible.

2. a) $q = m$ b) $p \times n$ c) square matrix i.e. $m = n$

3. Entries represent the total number of points obtained by each team

$$\mathbf{MN} = \begin{array}{c} P \\ Q \\ R \\ S \\ T \end{array} \begin{pmatrix} 5 \\ 3 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

4. a)

$$\mathbf{OA} = \begin{array}{c} \\ \text{June} \\ \text{July} \end{array} \begin{array}{c} \begin{array}{ccc} C & S & P \end{array} \\ \begin{pmatrix} 595 & 1080 & 265 \\ 620 & 1130 & 275 \end{pmatrix} \end{array}$$

The entries represent the total time needed for cutting, sewing and packing for the June (first row) and July (second row) order.

b) 45 days (of 24 hours)

c) 275 h

d) AO cannot be computed as (3×3) cannot be multiplied with (2×3)

5 a)

$$\begin{array}{c} \\ \\ \\ \text{from} \\ \\ \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array} \begin{array}{c} \text{to} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \end{array}$$

b) row totals: number of roads leaving a place/column totals number of roads entering into a place

c)

$$\mathbf{R}^2 = \begin{array}{c} \\ \\ \\ \text{from} \\ \\ \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array} \begin{array}{c} \text{to} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 5 & 5 \end{pmatrix} \end{array}$$

d&e) The entries in the first row represent the number of ways one can travel from A to A while passing through one other place (a two stage journey), the number of ways to make a two stage journey from A to B, and the last entry in the row, the number of two stage journeys from A to C. The other entries have similar meaning.

The row totals represent the total number of two stage journeys that can be made starting from A. The column totals represent the number of stage journeys that end in A.

7. a) $a = {}^-d$ b) $a^2 = {}^-bc$

   c) the matrices (18 in all) are of the format $\begin{pmatrix} a & b \\ -\dfrac{a^2}{b} & {}^-a \end{pmatrix} = \begin{pmatrix} 6 & b \\ \dfrac{{}^-36}{b} & {}^-6 \end{pmatrix}$

   where b is a factor of 36 i.e.

   ⁻1, 1, ⁻2, 2, ⁻3, 3, ⁻4, 4, 6, ⁻6, 9, ⁻9, 12, ⁻12, 18, ⁻18, 36, ⁻36.

8. a)

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ap + cq & bp + dq \\ ar + cs & br + ds \end{pmatrix}$$

   The product matrices are generally different.

   b)

$$(\text{AB})\text{C} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\right]\begin{pmatrix} k & l \\ m & n \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix} =$$

$$\begin{pmatrix} apk + brk + aqm + bsm & apl + brl + aqn + bsn \\ cpk + drk + cqm + dsm & cpl + drl + cqn + dsn \end{pmatrix}$$

$$\text{A}(\text{BC}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} pk + qm & pl + qn \\ rk + sm & rl + sn \end{pmatrix} =$$

$$\begin{pmatrix} apk + brk + aqm + bsm & apl + brl + aqn + bsn \\ cpk + drk + cqm + dsm & cpl + drl + cqn + dsn \end{pmatrix}$$

   Hence (AB)C = A(BC)

Self mark exercise 5

1 a)

$$\text{A} = \begin{array}{c} \begin{matrix} \text{P} & \text{Q} & \text{R} \end{matrix} \\ \begin{pmatrix} 1 & 2 & {}^-3 \\ 3 & {}^-3 & {}^-1 \end{pmatrix} \end{array}$$

   b) enlargement with centre O, scale factor ⁻ 2

   c) enlargement with centre O, scale factor k.

2. rotation about O, through ⁻90°

3. a) reflection in the x-axis, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & {}^-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   b) rotation about O through 90°, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & {}^-1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

   c) orthogonal projection on the y-axis, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

d) reflection in the line with equation y = 3, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & ^-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

4. a) rotation about O through 180° (or enlargement centre O, scale factor ⁻1)
   b) orthogonal projection on the x-axis
   c) stretch parallel to the x-axis (perpendicular to the y-axis) by factor 3
   d) stretch parallel to the y-axis (perpendicular to the x-axis) by a factor ⁻2

5. reflection in the line with equation $x + y = 6$

References

Hart, K., *Children's Understanding of Mathematics 11 - 16*, 1981, ISBN 071 953 772X

NCTM, *Historical Topics for the Mathematics Classroom*, 1989, ISBN 087 353 2813

Additional References

In preparing the materials included in this module we have borrowed ideas extensively from other sources and in some cases used activities almost intact as examples of good practice. As we have been using several of the ideas, included in this module, in teacher training over the past five years the original source of the ideas cannot be traced in some cases. The main sources are listed below.

Hart, K., *Children's Understanding of Mathematics 11 - 16*, 1981, ISBN 071 953 772X

NCTM, *Geometry in the Middle Grade*, 1992, ISBN 087 353 3232

NCTM, *Learning and Teaching Geometry*, 1982, ISBN 087 353 835X

NCTM, *Geometry from Multiple Perspective*, 1991, ISBN 087 353 3305

NCTM, *Assessment in the Mathematics Classroom*, 1993 yearbook.

Shell Centre, *Be a Paper Engineer*, ISBN 058 203 4906

Further reading

The Maths in Action book series are for use in the classroom using a constructivist, activity based approach, including problem solving, investigations, games and challenges in line with the ideas in this module. Material in this module has been taken from the Maths in Action books.

OUP/Educational Book Service, Gaborone, *Maths in Action Book 1*, ISBN 019 571776 7

OUP/Educational Book Service, Gaborone, *Maths in Action Book 2*, ISBN 019 .....

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 1*, ISBN 019

OUP/Educational Book Service, Gaborone, *Maths in Action Teacher's File Book 2*, ISBN 019

G Glossary

altitude	line segment in a triangle from a vertex perpendicular to the opposite side
analytical geometry	see *coordinate geometry*
centroid (triangle)	point of intersection of the three medians in a triangle
congruent shapes	two geometrical shapes are congruent if they are identical in shape (this include cases where one is the mirror image of the other) and size
coordinate geometry	a method in geometry in which lines, curves, surfaces, etc., are represented by equations and or inequalities using the coordinate system
coordinate system	a system for locating points in a plane or in space by using reference lines or points
dilatation	transformation of the plane onto itself where $(x, y) \Rightarrow (cx, cy)$. c is called the scale factor.
enlargement	see *dilatation*
figure (geometric)	a combination of lines, points , curves
half line	a straight line extending indefinitely in one direction from a fixed point
gradient	tangent of the angle a line makes with the positive x-axis
height (triangle)	length of the altitude in a triangle
image	if a shape S is mapped under a transformation T unto S′. S′ is called the image of S under the transformation T
latitude	angular distance of a point on the earth's surface measured from the equator along the meridian passing through that point
line	set of points (x, y) satisfying the equation $ax + by + c = 0$, where a, b and c are real numbers and a and b not both equal to 0
linear equation	linear equation in two variables x and y is an equation of the form $ax + by + c = 0$
line segment	the part of the line between and including P and Q, where P and Q are two points on a straight line

longitude	angular distance of a point on the earth's surface measured along the equator between the prime (zero or Greenwich) meridian and the meridian through the point
matrix	rectangular array of entries i.e. an arrangements with the entries displayed in rows and columns.
median	line segment in a triangle from vertex to the midpoint of the opposite side
mirror line	the line in which a shape is reflected
order of matrix	number of rows times number of columns
original shape	if a shape S is mapped under a transformation T unto S′. S is called the original shape
parallel lines	straight lines with the same gradient
perpendicular lines	lines that intersect at right angles
ray	see *half line*
reflection	if m is a line in the plane then the transformation mapping P onto P′ such that PP′ is perpendicular to m and PP′ is bisected by l is called a reflection
rotation	the transformation of the plane where one point C, centre of rotation, maps onto itself and each point P maps onto P′ such that CP = CP′ and $\angle$ PCP′ = θ, where θ is the angle of rotation
scale factor	in the transformation $(x, y) \Rightarrow (cx, cy)$ of the plane onto itself, c is called the scale factor
shape (geometric)	see *figure*
similar shapes	two geometrical shapes are similar if they have the same shape but not necessarily the same size.
slope	see *gradient*
transformation	is a one-to-one mapping from the plane onto itself. Each point S of the plane maps onto a unique point S′, and is the image of one and only one point.
translation	the transformation of the plane onto itself where P (x, y) maps onto P′$(x + a, y + b)$

Appendix 1: 1 cm square graph paper